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THE EFFECTS OF STRUCTURAL DIAGRAMS ON THE ACQUISITION OF KNOWLEDGE STRUCTURE AND PROBLEM-SOLVING PERFORMANCE IN MATHEMATICS

A THESIS

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By

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ABSTRACT

The literature reviewed shows that individual differences in problem-solving performance can be largely attributed to the organization of knowledge. In this study, an instructional means called the structural diagram was introduced into the instruction of a topic of school algebra -- arithmetic progressions -- with the hypothesis that a knowledge structure more similar to the standard one could be acquired. Methodology of cognitive science such as protocol analyses was adopted to investigate the knowledge structure and problem-solving behaviours.

The pilot study involved the development of test items for a written test and task-based interviews. Instructional materials were prepared with reference to the common mistakes revealed in pilot interviews. In the main study, two Form 3 classes from two schools were used, one as the treatment group, the other as control. Instructional method and mathematical reasoning ability were between-subjects variables, with mathematics achievement as a covariate. Problem-solving test scores were criterion measures. Knowledge structures acquired for a selected sample were inferred from protocols collected in task-based interviews. Their problem-solving performance was also observed through protocols.

It was found that the structural diagram could significantly improve students' problem-solving test score in routine problems, but not in nonroutine problems, when mathematics attainment was controlled. The overall test score also showed significant difference. Hypothesized interaction effects of instructional method and reasoning ability level were not found for all three problem-solving test scores.

Protocol analyses revealed that the inferred knowledge structures of the treatment group were characterized by better acquisition of formulas and procedures in handling routine problems but no significant difference was observed in conceptual understanding and the overall organization of knowledge items. In problem-solving performance, more frequent use of working forward or forward searching strategies and formal algebraic problem representations was observed in the treatment group. But, the quality of problem representations especially for harder problems did not show noticeable difference. In terms of qualitative comparisons, marginal interaction effects of the treatment and the reasoning ability level on the acquisition of knowledge structures, on the use of problem-solving strategies and on the type of problem representations employed were observed, the low ability group benefiting more from the treatment. The statistical and qualitative analyses yielded consistent findings, the latter providing clues to explain the former.

When knowledge structures and problem-solving performance were analysed in terms of good and poor performance classifications, the results were in general agreement with those of expert-novice studies. But backward reasoning and means-ends analysis were almost absent from strategies used by the subjects interviewed, contrary to the well-documented finding that novices tended to use general search strategies.

The results indicated that using structural diagrams would facilitate the acquisition of some basic problem-solving schemata. However, it might not be effective in fostering conceptual understanding. This was evidenced by the lack of significant differences in nonroutine problem-solving score and in the quality of problem representations between the two groups.

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CHAPTER I INTRODUCTION

Background and Problem of the Study

Problem-solving belongs to the highest level of human intellectual activities. In one form or another, it is considered as a desirable goal in most educational programmes. In view of the present rapid expansion of knowledge, it becomes increasingly important that students can utilize what they have learned to handle nonroutine situations and solve novel problems (Simon, 1980). To some, problem-solving in itself is also regarded as the best context for learning (Elshout, 1987).

In the past few decades, problem-solving has been an area of active study for researchers from various fields which include psychology, education, cognitive science and artificial intelligence. It has been said to be a very chaotic research area in psychology (Lester, 1980b; Resnick & Glaser, 1976). In the late 1950's, the information processing approach was introduced to psychological studies in general and problem-solving research in particular (Reitman, 1965; Newell & Simon, 1972). Much of the earlier information processing studies of problem-solving focused on artificial puzzles and toy problems to identify general problem-solving processes. These studies implied that to improve students' problem-solving performance, general domain-independent strategies and heuristics should be taught (Glaser, 1984; Resnick, 1985).

In the past ten years, problem-solving research has been conducted in the more semantically rich domains such as physics (Glaser, 1984; Gick, 1986). The former content-free approach gradually shifted toward a knowledge-based emphasis in understanding human intellectual performance. For instance, Goldstein and Papert (1977) said, " [T]here has been a shift in paradigm. The fundamental problem of understanding intelligence is not the identification of a few powerful techniques, but rather the question of how to represent large amounts of knowledge in a fashion that permits their effective use and interactions" (p.85).

Recent models of problem-solving in specific domains have focused on knowledge representations, schemata and mental models. Strategies specific to subject domains have been documented by cognitive scientists and educational psychologists (see e.g., Chi, Feltovich & Glaser, 1981; Larkin, 1981, 1983, 1985; Reif & Heller, 1982; Kintsch & Greeno, 1985; Heyworth, 1988). These studies have clearly demonstrated that the structure of the specific knowledge base determines the efficiency of the problem-solving processes (Glaser, 1984, 1987, 1988). An important direction of current research in instruction is to investigate how a body of knowledge is organized and represented by the learner at different stages of learning so that it can facilitate problem-solving and lead to competence in the subject matter (Resnick, 1985; Glaser, 1987).

In the field of mathematics education, since the pioneering works of Polya (1945, 1954, 1962, 1965), problem-solving has also occupied an important area of concern. During the 1970's, researchers devoted most of their attention to problem-solving and it is expected to continue for some time (Lester, 1980b). Two questions have been raised (Lester, 1983). The first one is whether any existing psychological theories are adequate to explain mathematics problem-solving or special theories have to be developed. The second one queries whether certain fundamental mathematics problem-solving skills and processes belong to the domain of mathematics exclusively. While these two questions cannot be answered thoroughly at present, it illustrates the trend that there is a growing interest in theory-based research among mathematics educators. Furthermore, clinical studies of mental processes and nonexperimental procedures employing qualitative analysis have been more widely accepted among them too (Lester, 1983).

So following the knowledge-based emphasis in current problem-solving research, it is natural to ask: in the domain of mathematics, what is the precise relationship between the knowledge structure of some specific content possessed by a student and his problem-solving performance? What is the role of general strategies in such performance? How can we effect a well-organized knowledge structure by instructional means? Will such instructional means significantly affect the problem-solving performance? These are the problems the present study is scheduled to explore.

Purpose of the Study

The purpose of the present study is twofold. Firstly, it aims at investigating the effects of an instructional means known as structural diagrams on students' acquisition of the knowledge structure and problem-solving performance in a topic of secondary school mathematics. Secondly, through this direct manipulation, the relationships between the knowledge structure and problem-solving processes can also be explored.

More precisely, within the chosen topic for instruction, the following questions will be addressed:

1. What are the observed individual differences in students' knowledge structure?
2. What are the observed individual differences in students' problem-solving performance?
3. In what ways does the use of structural diagrams affect the acquisition of knowledge structure? Is it effective in fostering a more well-organized knowledge structure?
4. In what ways does the use of structural diagrams affect the problem-solving performance of students? Will it affect the problem-solving strategies employed by the students?
5. What are the observed relationships between the knowledge structure and the problem-solving performance?

Significance of the Study

Regarding the significance of this study, several aspects can be noted. The most obvious implication concerns with the teaching practice in mathematics. While a variety of teaching strategies can be employed to teach mathematics (see e.g., Bell, 1978), structural diagrams are simple to use and are hypothesized to be effective in highlighting relationships and procedural rules. If it is demonstrated to be so, it can be readily incorporated into the present teaching practice without much difficulty.

Whereas problem-solving research in mathematics can be described as abundant, this kind of detailed qualitative analysis of knowledge structure is not so widely studied compared with those in physics (Romberg & Carpenter, 1986). As a well-organized knowledge structure should be a desirable goal of mathematics instruction (Resnick & Ford, 1981), studies in this direction could make contributions in providing an empirical base for formulating instructional theories. In fact, Glaser (1987) has pointed out that current instructional research should explore the knowledge organization at different stages of learning that leads to competence in various subject domains.

Although the role of general strategies in mathematics learning has been controversial in the current trend of emphasizing knowledge-based strategies, it appears that general strategies do play a special part in mathematics problem-solving as many mathematicians and theorists believe (see e.g., Polya, 1962, 1965; Greeno, 1980c; Schoenfeld, 1979, 1985b, in press; Goldin, 1985). For instance, Schoenfeld (1985b) argued that "[P]hysics is organized and taught according to deep structure. This consonance of principles and instructional organization does not hold in mathematics, where it sometimes appears that the underlying principles of solution and subject matter presentation are essentially independent of each other" (p.250). While the present study cannot be expected to do much about this controversy, it may nevertheless provide some empirical information for further consideration of the role of general strategies in teaching mathematics problem-solving.

CHAPTER II REVIEW OF RELATED LITERATURE

Theoretical Background of Problem-Solving Research

The term "problem" has a broad and indefinite scope of use among researchers and labels a variety of tasks ranging from riddles, mathematical puzzles to situations in real life. Polya (1962) regarded solving a problem as "finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately attainable" (p.v). Although Polya gave this definition in the context of mathematics problem-solving, it seems to be widely applicable. Newell et al. (1972) considered that "a person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it" (p.72). It is generally agreed that a problem can be defined in terms of three characteristics, namely, givens, goals, and obstacles (Mayer, 1983).

The experimental study of problem-solving was begun by Gestalt psychologists, especially by Duncker (1945), Köhler (1927) and Wertheimer (1945). However, such research was "sporadic" until a new theory of problem-solving based on concepts of information processing and computer programming was introduced in the late 50's (Green, 1966). Several psychological schools of thought have attempted to explain problem-solving phenomena. Three major psychological approaches can be identified (Green, 1966; Greeno, 1978a; Mayer, 1983; Newell, Simon & Shaw, 1958). These are the Gestalt approach, the behavioural approach and the information processing approach.

These approaches have their roots in different psychological traditions so that each of them specifies its own set of important concepts and variables which correspond to particular aspects of problem-solving phenomena distinct from the others. Gestaltists considered problem-solving as the restructuring of components of a problem in a new way (see e.g., Wertheimer, 1945). Associationists or behaviourists treated it primarily as a trial-and-error activity (see e.g., Skinner, 1966). The information processing approach conceptualized the human mind as an information processing system capable of manipulating symbols, switching methods and representations, and making decisions at different stages of processing (Newell et al., 1972).

Apart from these three approaches, Rowe (1985) also reviewed the close connection of problem-solving research with psychometric studies, in which problem-solving behaviours were linked to intelligence factors through correlation models (see e.g., Lee, 1981; Liu, 1982; Rowe, 1985).

<u>Information Processing Theory of Human Problem-Solving</u>

The basic conceptualization of the information processing view is that human behaviours are the result of the operation of several separable subsystems of information processing mechanisms: sensory-perceptual systems, central processing system (thinking and problem-solving), memory system and response systems (motor control) (Norman, 1981; Palmer & Kimchi, 1986). This theory asserts specifically that thinking and problem-solving can be explained by means of information processing. Problem-solving behaviour is described as an interaction between an information processing system (the problem solver) and a task environment (Newell et al., 1972; Simon, 1978). In approaching the task, the problem solver represents the situation as a problem space which is his own way of viewing the task environment. These three components -- information processing system, task environment, and problem space -- establish the framework for the problem-solving behaviour (Newell et al., 1972). Figure 1 shows such a model.

Problem-solving process. Problem-solving can be considered as composed of two major stages: an understanding process that generates a problem space and a solving process that searches for solution within the problem space (Greeno, 1978a, 1980c; Simon, 1978). However, this order is not invariable. Rather, frequent alternation between the two stages is observed (Simon, 1978). When a problem is understood, the subject has constructed a structural network that represents the relationships among the components in the problem (Greeno, 1973). The solving process can be represented as the completion of a relational network among the nodes so that the goal state can be eventually connected to the given state. This kind of representation was informally used at an earlier time by Polya in his discussion of solving

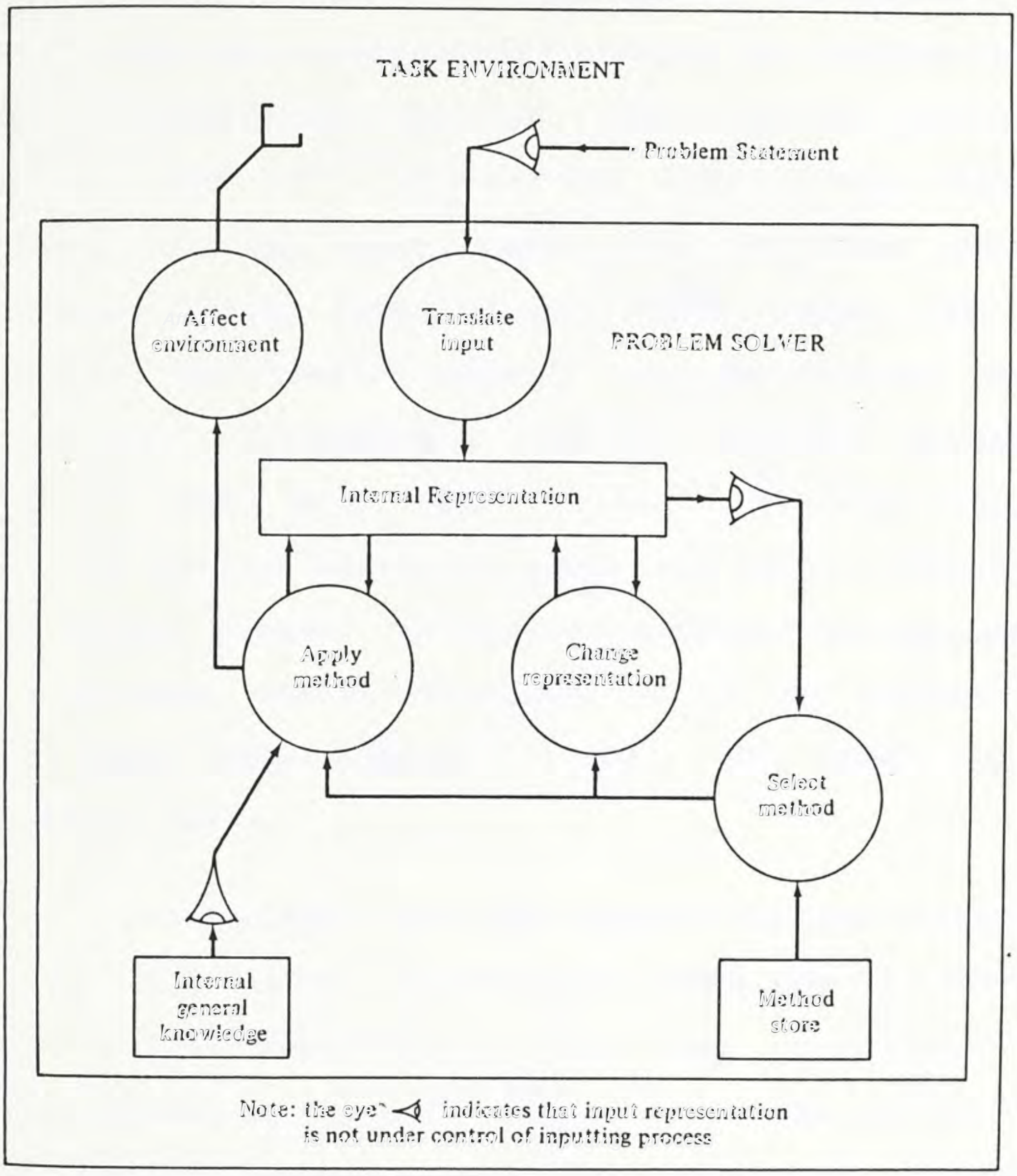

Figure 1. Information processing model of problem-solving (reproduced from Newell et al., 1972, p.89).

mathematics problems (see e.g., Polya, 1965).

Searching for solutions. In this formulation, problem-solving is thought of as a search through the problem space for a solution path. Search can be conducted by random search and heuristic search. When a familiar situation is encountered, well-formulated systematic procedures for solution called algorithms may be applied.

Following the tradition of experimental psychology, most of early information processing studies in problem-solving dealt with artificial puzzles, toy problems and simple tasks so that the interaction of pre-experimental knowledge with the experimental task variables could be eliminated (Gick, 1986; Glaser, 1987; Spada, 1987). As the search for solution usually did not interact with a rich storage of knowledge, general heuristic strategies were sufficient. Thus, these research studies aimed at describing general heuristic processes such as means-ends analysis, and subgoal formulation without considering the rich knowledge resources possessed by the person (see e.g., Greeno, 1978a; Hayes & Simon, 1977; Newell et al., 1972; Reed, 1977).

Since the time when psychologists turned to problems in the more semantically rich domains such as physics and mathematics, it has been recognized that general strategies have to adapt to a large structure of domain-specific knowledge and that a thorough analysis of this knowledge is a necessary pre-condition to understanding one's problem-solving performance in that domain (Glaser, 1987; Spada, 1987). In such knowledge-based problem-solving, we have to resort to such notions as "schema", "mental model", "knowledge representation" and "knowledge organization" for more comprehensive theories (Gick, 1986; Glaser, 1984, 1987)(for such studies, see e.g., Gentner & Gentner, 1983; Heyworth, 1988; Hinsley, Hayes et al., 1977; Kintsch et al., 1985; Mayer, 1985, 1987; Reif et al., 1982; Schoenfeld, 1985b).

Role of Knowledge in Problem-Solving

Modern learning theory recognizes that a person learning facts, concepts, rules and theories has acquired a large collection of knowledge structures (Greeno, 1980a). Research on expert and novice problem-solving has shown that the relations between the structure of the knowledge base and problem-solving processes are mediated through the quality of the problem representation (Glaser, 1984, 1986). As a person constructs the problem representation on the basis of his domain-specific knowledge, the nature of this knowledge organization influences the quality, completeness and coherence of the representation (Glaser, 1984, 1986; Greeno, 1977). These factors in turn determine the efficiency of problem-solving and learning.

Problem-solving in experts and novices. The work on problem-solving in experts and novices has been extensively carried out in a variety of subject-matter domains including chess, physics, mathematics, computer programming and the performance of electronic technicians (see e.g., Chi et al., 1981; Chi, Glaser & Rees, 1982; de Groot, 1966; Glaser, Lesgold & Lejoie, 1987 ;Jeffries, Turner, Polson & Atwood, 1981; Larkin, McDermott, Simon & Simon, 1980;; Schoenfeld & Herrmann, 1982; Simon & Simon, 1978). Fairly consistent findings have been obtained.

These research studies have indicated that experts' problem representations are qualitatively different from that of the novices. Experts possess specific problem-solving strategies relevant to problems in their domain of expertise so that they seldom need to use general search processes. Furthermore, the rich knowledge base of experts enables them to use more sophisticated procedures successfully. On the other hand, such strategies are poorly used by novices. They may sometimes lack the necessary strategies for problems and have to resort to general search strategies which may be inadequate for successful solution. Experts, driven by appropriate specific strategies, usually work forward while the novices, relying on such general strategies as means-ends analysis, work backwards in determining subgoals.

Problem categorization and representation. In their widely quoted study on physics problem-solving, Chi et al. (1981) found that novices' representations were organized around the literal objects and surface features given explicitly in the problem statements. Experts approached the problems on physics concepts and principles. Protocol analysis suggested that experts' knowledge was organized around principles and procedural knowledge about their applicability. Their declarative information was tied to conditions and procedures for its use and hence their ready access to solution procedures based on the underlying principles. Novices' organization of knowledge contained many surface features in forming their network. This kind of differential categorization of problems in terms of "surface structure" and "deep structure" for novices and experts respectively was similarly observed for mathematics problems in the study of Schoenfeld et al. (1982).

Significance of expert-novice studies. As these expert-novice studies have indicated, the problem-solving difficulties of novices can be attributed largely to the organization of their knowledge base rather than to the limitations in their general processing abilities (Glaser, 1986). Human ability in managing complex tasks may thus be due more to fine representations than to complex computational routines (Keil, 1984). Of course, metacognitive abilities possessed by experts should not be underestimated as these can be particularly useful in unfamiliar situations (Glaser, 1986).

However, based on the literature reviewed, the instructional implications may not be as obvious as it seems. While these observations clearly describe the differences between experts and novices, they do not indicate nor give sufficient hints for the mechanisms in the acquisition of expertise. (To be fair, most of these studies did not make claims on this.) These studies do not mean that the poor knowledge organization of novices are to be attributed to poorly taught courses. The expert's knowledge structure cannot probably be acquired by direct instruction. Rather, the expertise has been gained through many years of learning and problem-solving experience. Furthermore, the present groups of experts under study were certainly themselves novices many years ago. But, is it possible that they were previously groups of novices with special aptitudes in their particular domains? Only longitudinal studies may provide further insight to the question. Thus, the present expert-novice studies do not have immediate implications for the content of instruction.

Nevertheless, as far as instructional research is concerned, these studies imply that innovative instructional means have to be invented by considering the factors that may facilitate the organization of knowledge presented and the re-organization of learners' existing knowledge and to be subsequently tested through empirical studies. On the other hand, expert-novice studies certainly highlight the significance of knowledge organization in assessing learning outcomes. As an indication of the level of acquired expertise, the state of knowledge organization can give diagnostic data for the assessment of the learner as well as the evaluation of an instructional programme.

Schema Theory

The notion of schema. The theoretical concept "schema" is a useful notion in explaining how structured knowledge facilitates problem-solving (Chi et al., 1981; Glaser, 1984, 1986, 1987, 1988). The present use of "schema" as a formalism for representing knowledge has come from research of artificial intelligence where it plays an important part in organizing large data bases. In fact, Rumelhart (1980) has regarded it as "the building block of cognition", the basic element upon which information processing depends. However, since the nature of the human information processing mechanisms is not fully understood, the notion of schema may be ambiguous (Glaser, 1987; Rumelhart, 1980). The psychological significance of this construct is still a question (Anderson, 1985). Schema theory attempts to describe how knowledge acquired is organized and represented and how these structures can facilitate the use of knowledge in different ways.

A schema can be defined as "a modifiable information structure that represents generic concepts stored in the memory" (Glaser, 1984). It represents the information available in our experiences and the usual sequences of events. New situations are interpreted through these schemata which integrate new information with prior knowledge. A relevant schema is activated if incoming information can fit into a sufficient number of variables or "slots" of that schema. An active schema may then guide and seek information to fill up more slots, constructing a more complete interpretation of the event encountered. If it fails to account for certain aspects of the situation, it may be adopted temporarily, rejected, replaced or modified (Glaser, 1984, 1986, 1988; Rumelhart, 1980).

Schema and problem categorization. According to Silver (1982a), the study of Hinsley et al. (1977) was the first to make an explicit link between schema theory and mathematical problem-solving. They used algebra word problems to trace the process of comprehension. It was found that college students could categorize word problems into groups of related problems. This categorization could be given after hearing only the first few words. While Hinsley et al. were primarily interested in distinguishing the two different approaches in comprehension, namely the "text grammar" approach and the "schema" approach, their general conclusion also established the fact that subjects possessed schemata for standard algebra problems and that these schemata directed the encoding and retrieval of information during problem-solving.

Two points concerning their study can be raised. To show that subjects could categorize problems very early in the course of reading, problems were read to them in parts and subjects were prompted to categorize these problems at each break. This procedure may be criticised as unnatural. Although the experiment showed that they could do it, what might happen during uninterrupted reading of problems remained unknown. Comparatively speaking, the method adopted by Chi et al. (1981), allowing the subjects to read the problem freely and think aloud simultaneously, would provide more decisive evidence in this respect. Secondly, Hinsley et al. also claimed that subjects could use schemata in solving very "non-standard" problems. This statement should be taken carefully because their so-called non-standard problems were actually very standard algebra word problems with some words directly replaced by nonsense words. These "new" problems presented difficulty not in terms of mathematical sophistication but language comprehension rather.

Schema and problem representation. Recent research has established that problem representations are formed essentially through the action of existing schemata (Glaser, 1987, Rumelhart, 1980)(see also e.g., Chi et al., 1981; Hinsley et al., 1977; Kintsch et al., 1985). A familiar problem readily activates an appropriate schema which can solve the problem. Otherwise, schemata containing more general strategies may be triggered instead. The slots of a schema determine which features of the problem are going to be incorporated into the representation. Features that do not fit are ignored. Once the problem is represented, the set of activated schemata may guide and control information processing. When the relevant conceptual and procedural knowledge are contained in these schemata, the problem can be solved. Therefore, effective problem-solvers certainly possess a rich collection of schemata that correspond to prototypical structures of problems (Glaser, 1984, 1986). In the light of schema theory, the results of expert-novice studies may be likewise interpreted. Conversely, these studies also provide empirical support for this "schema" formulation.

Problem-solving process: An integrated model. While general problem-solving strategies and heuristics are heavily used in puzzles and toy problems, schema theory focuses more on the role of knowledge. An integration of these studies can provide a more elaborate model of the problem-solving phenomena through these two kinds of problem-solving strategies. Figure 2 shows such a model (Gick, 1986).

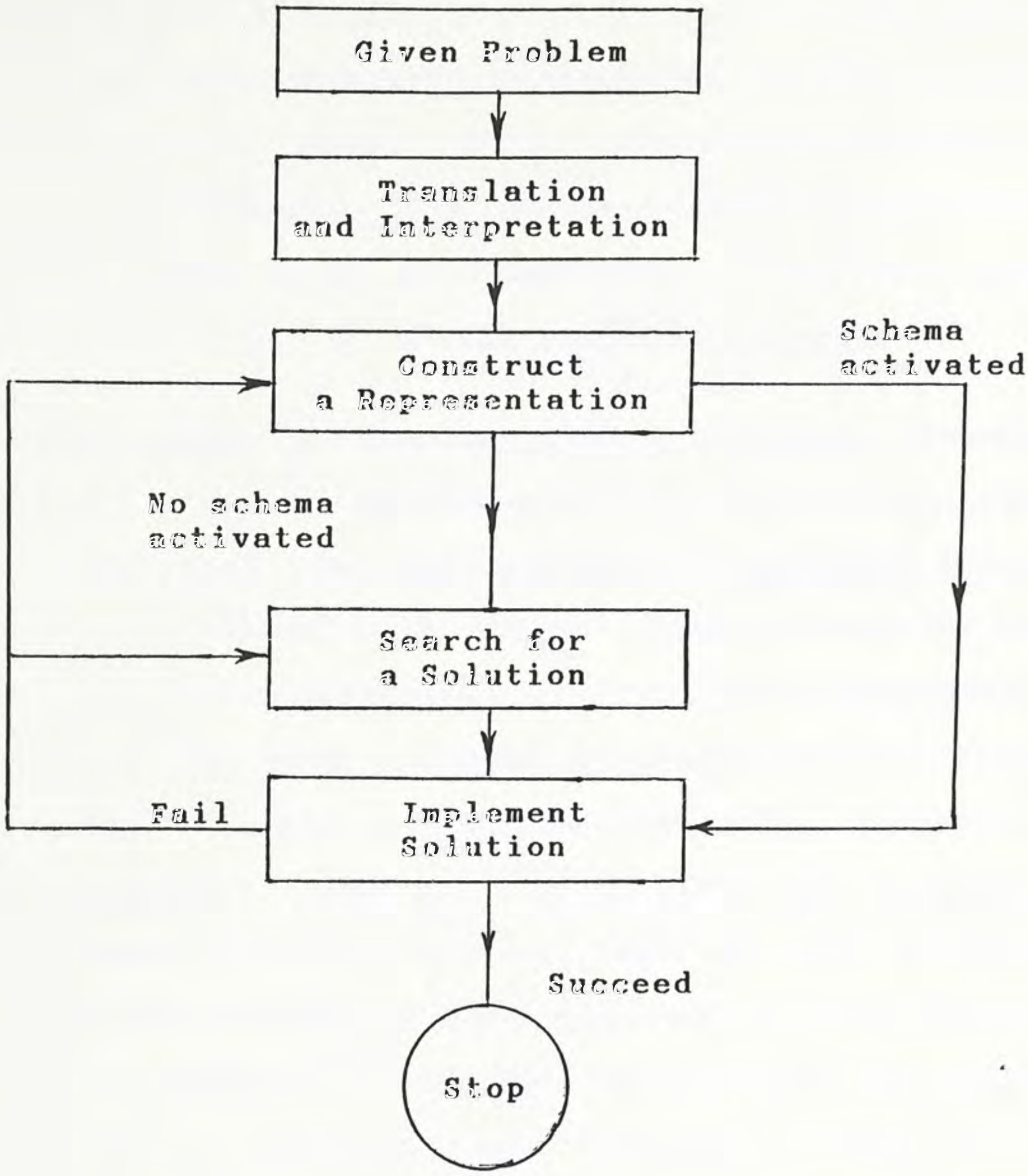

Figure 2. The problem-solving process (adapted from Gick, 1986).

When a problem can be solved by the activation of appropriate schemata, the problem-solving process is schema-driven, with little search strategies required. These strategies can be domain-specific or general. Otherwise, the problem-solver has to resort to some search strategies such as means-ends analysis, planning or analogical reasoning. Heuristics introduced by Polya (e.g., 1945) in mathematics problem-solving belong to this category. Other general strategies called metacognitive

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strategies may also be necessary in monitoring the application of methods and procedures during problem-solving (Flavell, 1976). When a person acquires more knowledge in a domain, his problem-solving behaviour in that domain tends to be more schema-driven and corresponds to more frequent use of domain-specific strategies.

Acquisition of problem-solving schemata. Based on the review of this integrated model of problem-solving, it can be suggested that the learning of relevant schemata for solving problems in a subject domain would be quite essential. A few questions arise from this consideration. Firstly, how can such a schema be acquired? What factors can contribute to its acquisition? Secondly, in what ways can we implement these in instruction for a specific subject domain? Thirdly, observe that the more elementary form of schema-driven problem-solving is restricted to some sort of standard or stereotyped problems in a subject domain. This learning task usually occurs at the foundation stage in secondary schools. Would this kind of learning hinder the learner's development in the more flexible type of problem-solving? This last question would be particularly relevant to researchers of mathematics problem-solving in which the primary focus of learning and understanding mathematics is certainly not on solving standard problems.

Although we have discussed only the schema theory in detail, it must be stressed that psychologists hold widely different views regarding the representation of knowledge and the different forms of knowledge. There is much controversy over the memory structure and the mechanisms of knowledge acquisition either. However, one basic idea is shared. In order to account for the human abilities in storing, retrieving and using knowledge, psychologists find it necessary to conceive of the human memories as organized or structured (Anderson & Bower, 1973; Greeno, 1980a; Resnick et al., 1981; Anderson, 1985).

There are different theories or models for the human memory but all of them impose the condition of structures and associations on the knowledge stored in the long-term memory (or semantic memory). Instead of stating the exact form of the human memory, the central idea of their assumption of structuredness is that concepts can be assessed in structures so that related concepts can be used efficiently. For instance, schema theory is a theory about how knowledge is represented and how such representation facilitates the use of knowledge in certain ways (Rumelhart, 1980). As reviewed previously, many researchers have used the notion of schema as a hypothetical construct for interpreting their observations and extrapolating their findings.

Knowledge structure. The term "knowledge structure" has been used by semantic memory theorists to refer to the organization of information stored in the memory, i.e. the relationships between concepts, facts and propositions (Resnick et al., 1981). Many recent studies have focused on how people organize their knowledge. Expert-novice studies are some typical examples in which the knowledge structures of experts and novices are contrasted. For instance, expert-novice studies in physics have indicated that experts tend to organize their knowledge based on principles (e.g. Chi et al., 1981; Chi et al., 1982). Heyworth (1988) has investigated the knowledge organization of secondary school students in the topic of volumetric analysis and documented the different mental representations of concentration, dilution and neutralization as used by expert students and by novice students. Law (1988) has compared the knowledge structures about motion among junior and senior secondary school students, and noted the prominent features of 'naïve physics' appeared more often in the junior students.

This kind of knowledge organization has been studied more widely in the learning of science than in mathematics (Romberg et al., 1986). One example is Geeslin and Shavelson's (1975) study of the cognitive structure in the learning of probability and the effect of a programmed text on its acquisition. But, Geeslin et al. mainly used word association technique to assess familiarity with and relations between concepts. In their achievement test, items primarily tested comprehension of text materials rather than solving problems. Greeno (1976) conducted another study in representing the knowledge structures of students' understanding of fractions and simple plane geometry as conceptual networks and procedural networks. This study focused more on the performance of students in utilizing knowledge to solve problems in the topics concerned.

It is generally agreed that better understanding and the ability to apply one's concepts and skills depend on a better organization of acquired facts. Conversely, analysis of students' knowledge structures provides a means to evaluate their understanding of a topic.

Knowledge structure and problem-solving. In this conceptualization, information and concepts are established in specific relationship to one another and learning may consist of constructing new connections and integrating new items to the network. Referring back to the two stages of problem-solving (i.e. understanding and searching for solution), we can see how the knowledge structure may interact with the processes. The internal representation of the problem is constructed on the basis of information retrieved from the memory. The information given in a problem is then encoded in a way that matches elements in the subject's knowledge structure. Thus, a problem is expressed in the subject's own way. Greeno (1977) has suggested a reasonable hypothesis that when elements in the problem representation are connected strongly to other components of knowledge, the subject can readily answer interpretative questions about this structure and relate this structure to other situations. Expert-novice studies in the early 80's may be interpreted as giving this idea firm empirical support and in fact have elaborated it to a great extent.

The more elegant the knowledge structure is organized, the more probable a problem can be solved in that specific subject-matter domain. This idea of connectedness of a problem representation to the subject's knowledge base resembles the Gestalt psychologists' emphasis on understanding and insight. (But, Gestaltists held a distinct view on the nature of such organization.) Certainly, different representations may call up different concepts and procedures in the long-term memory (Resnick et al., 1981). Therefore, to facilitate problem-solving, instruction should ensure a well-structured knowledge and maximize the links between concepts and procedures (Glaser, 1984; Resnick et al., 1981; Spada, 1987).

Instruction to facilitate knowledge organization. To facilitate the acquisition of ideal knowledge structure, teachers are advised to teach well-organized materials and to encourage students to actively link up the ideas themselves (Heyworth, 1988). In the case of structured materials, several techniques can be employed: diagrams showing the hierarchy of key concepts as an introduction, advance organizers, and other appropriate diagrams such as time-sequence charts or flow-charts. Recent work on diagrammatic structures has demonstrated a better recall of information and it is also suggested that teachers should promote different ways of reorganizing knowledge rather than teach in terms of a single definite structure (Heyworth, 1988).

To achieve well-structured knowledge in mathematics, Resnick et al. (1981) suggested that the subject-matter structure should be revealed by the instruction and students should have opportunities to practise new procedures and concepts in different contexts. Greeno (1978c, 1977) emphasized learning mathematics with understanding for which three criteria of judgment were proposed, namely the internal coherence of the representation of the information, the degree of connectedness of the information to other pieces of existing knowledge, and the correspondence of the representation with the information to be understood. Based on his idea of understanding, instructional activities can presumably be evaluated in terms of these criteria too.

With the advance of technology, new developments promise innovation in instruction. Computer technology can provide an interactive environment for students to handle abstract mathematical ideas in some concrete and manipulable ways (Papert, 1980). For instance, Schoenfeld's (1988a) study of students confronting mathematical functions and graphs in a computer-based microworld has demonstrated connectedness and firmness in their conceptual understanding of these notions. It has been claimed that the ready manipulation available in the interactive environment can establish more links between the conceptual entities involved.

Content, Cognition and Individual Differences in Problem-Solving

Problem-solving research has been regarded as quite non-cumulative and separate studies focused on many different aspects of problem-solving (Lester, 1980b; Rowe, 1985). A wide variety of tasks and problem situations have been used, thus making comparison between studies difficult. In this respect, it is useful to classify the focus of these studies into three categories, namely, the content, the cognition, and the individual differences in human problem-solving (Rowe, 1985; Scandura, 1977). Any problem-solving research would deal with these three psychological aspects and their interactions to different extent. Any comprehensive model of human problem-solving should, as a minimum, involve specifications on these aspects.

Cognition. Cognition occupies the key role in understanding how a successful problem-solver can produce a solution. The major goal of the psychologists in the study of problem-solving is to investigate the nature of the underlying cognitive processes involved. Many different techniques have been employed to investigate these cognitive processes, ranging from direct verbalization of thoughts, clinical observation to experimental tests.

However, whereas psychologists intend to deal with the problem-solving process in its full complexity, their studies must involve problem situations which may be composed of specific problem-solving competencies and general cognitive processes in unknown proportion. This causes difficulties in the interpretation of results with the confounding problem of content and cognitive processes (Scandura, 1977). Furthermore, such factors vary to different extent among individual subjects even with the same task. With regard to such subtle effects of confounding variables of content, cognition and individual differences, psychologists are expected either to obtain only information concerning the general tendencies in problem-solving behaviour or to construct very specific model for an individual (Scandura, 1977).

Content. In the study of problem-solving, it is equally important to identify specific competencies that are necessary for solving particular classes of problems. Polya's (1962, 1965) study of mathematics problem-solving is a significant pioneering work in his field, providing an informal approach to the kinds of capabilities necessary for solving many mathematics problems.

Presumably owing to the pragmatic concerns of educational psychologists, they are relatively more active in this aspect of analysing content (Scandura, 1977). A variety of methods such as task analysis and content analysis are developed for precise formulation of tasks. Resnick et al. (1976, 1981) proposed task analysis as a strategy for linking psychological constructs to instructional concerns. They also explored its relevance to the domain of problem-solving, suggesting that task analysis, formally stated and empirically validated, would characterize the specific abilities in instructable terms so that problem-solving could be effected through instruction. Greeno (1980b) also emphasized the contribution of task analysis in the issue of teaching problem-solving strategies in geometry.

Individual differences. Individual differences in problem-solving ability are also important to a comprehensive understanding of the problem-solving phenomena. The causes of the observed differences have to be investigated. Recent research in developmental studies and problem-solving of experts and novices has indicated that a salient difference lies in the possession and utilization of an organized body of conceptual and procedural knowledge (Glaser, 1984, 1988). From the viewpoint of individual differences, research in human problem-solving involves the construction of appropriate measures of various cognitive abilities, yielding test measures that correlate with problem-solving ability. Such studies can furnish diagnostic information that can provide a prescription for educational intervention (Rowe, 1985).

The importance of all these three aspects of human problem-solving is generally acknowledged although an individual framework of study may emphasize one or another factor. A thorough understanding of problem-solving behaviour should certainly address each one of these aspects and their interaction. For this, Rowe (1985) made the following assertion:

> Any research concerned with the question of how people solve problems must be performed within a framework which takes into consideration variables from at least three major sources of expected variation, namely characteristics of the psychological mechanisms which operate during performance, individual differences, and the task. A process model of problem solving should be consistent in terms of all three domains, and should aim to account for interactions between these domains. (pp.15-16)

CHAPTER III FRAMEWORK OF STUDY

The literature reviewed in the last chapter sets the theoretical background for applying the cognitive science methodology to a study of problem-solving in the domain of mathematics. In this chapter, issues more specific to research in mathematical problem-solving are discussed.

Problem-Solving Research in Mathematics

Owing to the nature of mathematics learning, mathematics educators and teachers are particularly conscious of the problem-solving performance of students. According to Lester (1983), past problem-solving research in mathematics has not been conducted systematically and is largely atheoretical. Agreement is lack on what constitutes problem-solving, how performance is to be assessed, what tasks are suitable for research and what the key variables are that influence problem-solving behaviours (Lester, 1980b, 1983). It is observed that the methodology of cognitive science is now more widely adopted by researchers in this area of study.

Polya's (1945) well-known four phases for problem-solving have stimulated much thought among mathematicians and educators alike. Understanding the problem, devising a plan, carrying out the plan and looking back constitute a prescriptive model for learning to solve problems. However, it has not incorporated any person or task variables. Research generated by Polya's suggestion usually focused on instruction of heuristics (Lester, 1985).

Schoenfeld (1985b) offered a theoretical framework for analysing mathematical thinking or other higher-order thinking skills. It has been proposed that problem-solving behaviours in mathematics should be investigated in terms of four components:

1. cognitive resources -- the body of mathematical facts and procedures possessed by an individual
2. heuristics -- the general mathematical problem-solving strategies an individual can use in difficult situations
3. control -- the competency of an individual in making decisions on how to apply knowledge and procedures
4. belief system -- the set of understandings and perspectives regarding the nature of mathematics an individual holds

This framework was developed on the basis of much empirical research by Schoenfeld (and his colleagues) in the past decade using primarily the cognitive science methodologies (e.g. Schoenfeld, 1979, 1982, 1983a, 1983b, 1985a, 1985c, 1988b; Schoenfeld et al., 1982). In place of Polya's prescriptive model, it provides a theory that can interpret observed phenomena, make predictions, and guide future inquiry in problem-solving performance in terms of the four components which constitute the cognitive and metacognitive resources of an individual subject under study.

Conceptual and Procedural Knowledge in Mathematics

Knowledge is the foundation for problem-solving. The distinction between conceptual and procedural knowledge, though may not be taken as a strict classification scheme for all knowledge, certainly provides a perspective to interpret the success and failure in problem-solving performance (Hiebert & Lefevre, 1986). Conceptual knowledge is characterized as knowledge that is rich in relationships. It can be thought of as a network linking discrete pieces of information. An item of conceptual knowledge cannot be an isolated piece of information and must bear some relations to other units. Procedural knowledge consists of rules, algorithms or procedures to solve specific tasks. It is sequential in nature and may be structured with some procedures embedded in others as sub-procedures (Hiebert et al., 1986).

However, this distinction should not be made absolute since rules and structures at one level may be considered as concepts manipulated by rules at a higher-level (Lesh, 1985). It can also be argued that the acquisition of a concept contains the knowledge of a procedure to distinguish between examples and non-examples of that concept. Some common procedural errors may be associated with defects in conceptual understanding. Furthermore, understanding and interpretation in mathematical tasks must involve linkages between conceptual knowledge and procedural knowledge (Silver, 1986).

Effective instruction demands a thorough analysis of the knowledge required for a particular class of tasks (Greeno, 1980c; Resnick et al., 1981). It is also noticed that studies of error patterns can provide insight into the existing knowledge content and organization possessed by the learners. The systematic flaws in performance can be a rich source for understanding common misconceptions in conceptual and procedural knowledge of learners and provide diagnostic data for remedial instruction (e.g., Brown & Burton, 1978; Silver, 1986).

Individual Differences in Mathematics Problem-Solving

Traditionally, researchers in mathematics problem-solving have considered individual differences along two distinct directions (Lester, 1983; Threadgill-Sowder, 1985; Silver, 1985). One direction is to identify general aptitudes or individual characteristics which correlate with problem-solving performance. Research has been conducted to study factors such as mathematical background, verbal and reasoning ability, perseverance, tolerance for ambiguity, spatial ability, field independence, age and sex (Lester, 1980b, 1983). However, the information processing model would be promising to replace this psychometric tradition when such psychometric measurements can be identified with underlying cognitive processes (Threadgill-Sowder, 1985).

Another direction is to distinguish the "good" and the "poor" problem-solvers, and describe their characteristic approaches to problem-solving. The recent popular trend in expert-novice studies belongs to this category. For example, Silver (1979) observed that good problem-solvers had higher ability in perceiving the mathematical structures of problems. They could generalize their method of solution to problems of similar structures more readily. This trend of studies have also shifted away from identifying characteristics unrelated to cognitive behaviours toward those associated with specific mental processes (Lester, 1983).

Briars (1983) has analysed mathematical ability in terms of an information processing model. Three possible sources of individual differences in mathematical ability have been identified: differences in basic information processing skills, content knowledge, and metacognition concerning mathematics. Many studies have investigated the differences in content knowledge (e.g. Hinsley et al., 1977; Schoenfeld et al., 1982; Silver, 1979). The general conclusion is that able mathematics students organize their knowledge differently and they have a rich network of knowledge structure reflecting the structure of mathematics. But empirical studies in the other two areas of mathematical ability are comparatively few (Briars, 1983).

Instruction and Problem-Solving in Mathematics

What kind of teaching would facilitate students' problem-solving performance? Instructional experiments which aimed at teaching explicitly some problem-solving strategies and heuristics are commonly found in the research literature of mathematics education. Effectiveness of such schemes has been demonstrated (see e.g., Schoenfeld, 1979, 1982; Vos, 1976).

Recent studies on problem perception and categorization have indicated that students tend to acquire schemata as templates for understanding problems and such schemata would influence what information students would look for in problems (e.g., Hinsley et al., 1977; Silver, 1979, 1982b). This conclusion has led to a kind of "schema training" in which practice is given to recognition of problem types (e.g., Mayer, 1985, 1987). However, this instructional suggestion may be doubted to be the opposite of teaching problem-solving which should aim at new problems (e.g., Sowder, 1985).

The relevance of expert-novice studies to instruction similarly has led to different opinions. It has been suggested that in general, science instruction should be based on detailed cognitive analysis of the knowledge required for solving problems and such knowledge should be thoroughly taught (Glaser, 1988; Heller & Hungate, 1985; Resnick, 1985). Romberg et al. (1986) proposed that experts' analysis of conceptual maps could provide a framework for organizing instruction to emphasize linkages between related concepts. But would it be the same case with mathematics learning? Based on the success of a prescriptive model of teaching problem-solving in mechanics by Reif et al. (1982) which incorporated the descriptive theories of expert performance, Heller et al. (1985) suggested that this should also be true with mathematics.

In response to these, Goldin (1985) has argued that science and mathematics learning are different. In mathematics problem-solving, exploratory activities, trial and error, and heuristics are very essential, but these would be undermined if the instructional focus is on the knowledge content. Silver (1985) doubted these implications by suggesting that the path from novice to expert might be discontinuous and some novices might be only capable of becoming "highly experienced novices". Schoenfeld (1983b, 1985b, 1985c) stressed specifically the difference in metacognitive skills between experts and novices in the domain of mathematics. Greeno (1978b, 1980c) agreed that specific knowledge should be necessary but doubted it to be too mechanical in some cases and still maintained that more general concepts and procedures would lead to greater transfer.

Thus far, in spite of the shared goal of problem-solving in teaching mathematics, there has still been little agreement, if not contradicting views, on the best ways of instruction to enhance it.

The Present Study: An Instructional Experiment focused on the Knowledge Base

The literature reviewed shows that the role of the knowledge structure in problem-solving is indisputable. Individual differences in problem-solving performance can be largely attributed to the differences in the organization of knowledge which directly influences its utilization. What instructional strategies should be employed to teach this body of knowledge is an important research question that follows. Factors facilitating the acquisition of knowledge and development of expertise are waiting to be explored. On the other hand, researchers in mathematics problem-solving also put emphasis on the role of general strategies in mathematics.

Among the instructional experiments conducted in the area of mathematics problem-solving, evaluation of courses on general strategies and heuristics has been very common. Another line of attack stimulated by recent research has been the study of acquisition of a few schemata for solving standard problems through repeated practice or drilling (e.g. Mayer, 1987; Sweller, Mawer & Ward, 1983). The element of understanding may be ignored and the content and context of learning may be quite different from that in the usual classroom.

The present study introduces the structural diagram as an instructional means to expose the relationships between concepts, highlight the use of procedures and analyse the structure of mathematical problems. The content for the present instruction, though still limited in scope, resembles the usual material in the curriculum. In this respect, it contrasts with the kind of experiments which have only focused on a few schemata.

A structural diagram can show the mathematical relationships between concepts and rules in a subject area or between components in a problem. Polya (1965, chap. 7) used it to illustrate the solution paths of problems, though not giving it a specific name. This diagrammatic technique has also been employed by Anderson, Boyle and Reiser (1985) in their intelligent tutoring system which displayed on-screen the structures of geometry proofs in similar forms known as proof graphs. Anderson et al. claimed that this facility helped students manage working memory load so that learning from the problems could be enhanced.

As an instructional technique, the structural diagram is hypothesized to have the following functions:

1. facilitating the problem representation
2. enhancing qualitative analysis of problems
3. facilitating the search of solution methods
4. revealing the problem structure
5. emphasizing the procedural relations between the mathematical entities

It is hypothesized that through the emphasis in linkages between variables, a better understanding of a topic can be achieved and hence a more well-organized knowledge structure can be acquired. According to schema theory, these functions would also facilitate the building up of problem-solving schemata in the subject area. In view of the results of expert-novice studies, the instructional effect on problem-solving is hypothesized to mediate through the knowledge structure.

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In the present instructional experiment, the effects of the structural diagram on the knowledge structures acquired and consequently on the problem-solving performance would be studied. The relationships between the knowledge structure and the problem-solving performance could also be explored. In the following sections, issues pertaining to the instrumentation are discussed.

<u>Assessment of Problem-Solving Performance in Mathematics</u>

The instruments employed by the majority of researchers in mathematics problem-solving are of two types: (i) paper-and-pencil tests, and (ii) protocol analysis based on "thinking aloud" or retrospection (Lester, 1980b; Schoenfeld, 1985b). In the past, researchers tended to rely on quantitative measures of problem-solving performance (Lester, 1983). Many of the written tests used product measures rather than process measures (Schoenfeld, 1982, 1985b).

<u>Written tests</u>. Written tests have the obvious advantage of easy administration and a large number of subjects can be assessed within a relatively short period of time. However, most paper-and-pencil tests for mathematics problem-solving are unreliable and consist of routine problems only (Lester, 1980b). The choice-type test may tend to restrain the subjects' response. The performance data from written tests are normally handled by factor analysis and suitable statistical procedures. This approach cannot give indication of the variety of

procedures employed by the subjects (Schoen & Oehmke, 1980; Schoenfeld, 1985b). In view of these shortcomings, current attempts in written test construction have focused more on the types of strategies used, and the structure of the problem-solving process (see e.g., Collis, Romberg & Jurdak, 1986; Malone, Douglas, Kissane & Mortlock, 1980; Schoen et al., 1980; Schoenfeld, 1982)

<u>Protocols and the "thinking aloud" technique</u>. Another data-collection technique is based on protocols recorded when the subject solves problems. The "thinking aloud" procedure is most commonly adopted. Subjects are required to verbalize their ideas and actions during the problem-solving process. This is either audiotaped or videotaped. The recordings are then coded according to some specific coding schemes of classifying the subject's behaviours.

Regarding the "thinking aloud" technique, Ericsson and Simon (1980) presented an information processing model of how subjects, when required to think aloud, verbalized information that were processed in the short-term memory. Based on empirical studies of their model, Ericsson et al. concluded that verbalizing information would not affect cognitive processes unless the subject was instructed to verbalize those that would not be attended to otherwise. There are critics of the method. For instance, Lester (1980a, 1983) maintained that thinking aloud would affect the performance, and the problem-solver might tend to talk only about those moves he thought safe or correct. The procedure might also depend very much on the subject's linguistic development.

Introspection and retrospection are two other related techniques. Introspection differs from "thinking aloud" in that subjects are required to analyse their own thinking during the process. Retrospection requires such analysis after the problem is solved and the interviewer may ask the subject questions about the observed behaviours. Both procedures may suffer the danger of rationalization (Rowe, 1985). In retrospection, the subject may mix current knowledge with past knowledge too (Newell et al., 1972).

Protocol analysis. Following Kilpatrick's (1967) pioneering approach in coding protocols for statistical analyses, many researchers have developed detailed schemes for coding protocols in specific studies (see e.g., Ki, 1986; Liu, 1982; Rowe, 1985). The usual results are frequency counts of certain types of strategies used by the subjects and their correlation with other task or subject variables. This kind of quasi-qualitative analysis may not provide much insight to the nature of problem-solving behaviours (Lester, 1983).

Qualitative protocol analysis is becoming more popular currently. In fact, many expert-novice studies have employed this approach. Lester (1983) suggested that multiple criteria for performance should be employed in both qualitative and quantitative analysis, depending on the nature of the study. Schoenfeld (1983b, 1985b) has introduced another framework of macroscopic analysis of protocols of solving mathematics problems, concentrating on the switching of procedures that manifests metacognitive skills.

Techniques for Assessment of Knowledge Structure

Whereas the ideal knowledge structure can be obtained by a rational analysis of the subject-matter, the knowledge structures possessed by people can only be inferred from their overt behaviours. Various techniques have been developed by cognitive psychologists. The following have been employed: measurement of response latencies, study of written answers and error patterns, analysis of thinking-aloud protocols, measures of association among words, task-based interviews, and conventional experiments based on hypotheses generated from the above techniques (Chi et al., 1981; Geeslin et al., 1975; Heyworth, 1988; Resnick et al., 1981; Romberg et al., 1986; Stewart, 1980; Thro, 1978). However, some techniques may seem to be rather ad hoc and strategies have usually been designed for specific studies (Resnick et al., 1981).

It may be added that computer technology has also been applied recently in probing people's ideas and understanding of specific subject-matter (see e.g., Law, 1988; Schoenfeld, 1988a). Through the interactions between the computer system and the subject, conceptual structures of the subject can be revealed.

Task-based interview. Research studies on human thought processes and mental representations have adopted task-based interviews as the primary methodology to collect data which reflect people's knowledge structures and mental processes (Heyworth, 1988). For instance, the studies of Chi et al. (1981), Gentner et al. (1983), and Schoenfeld et al. (1982) all employed this technique. While the validity of verbal data collected during such interviews has been generally acknowledged among cognitive scientists, the information can only provide some clues to the internal processes and representations which have to be inferred from such data.

As task-based interviews were also conducted in the present study to investigate subjects' knowledge structures of the subject-matter, this technique will be discussed in more detail. In general, a subject invited to a task-based interview is asked questions which probe conceptual understanding or test problem-solving in a specific topic. The subject is normally requested to think aloud during problem-solving. The interviewer starts with some introductory conversation to put the subject at ease. Some practice questions to think aloud may be necessary. When it comes to the actual tasks, the interviewer only speaks to facilitate the subject's verbalization and should avoid any remarks that may direct or distract the subject. After finishing the tasks, the interviewer may need to ask questions to clarify some observations. The interview is usually audiotaped for subsequent analysis.

Notice that carrying out actual interviews may not be as simple and direct as it seems. Firstly, some subjects may feel so nervous that they cannot do much verbalization and fail to provide the necessary data. Some just don't have the habit to talk freely. This difficulty would be particularly serious with pupils of junior forms. Secondly, the interviewer may sometimes find it extremely difficult if not impossible to play the role of an observer and not to interfere especially when the subject explicitly requests for hints or comments.

CHAPTER IV METHOD

Definitions

The terms that are used in the present study are defined below. Some of these terms are particularly useful in describing the problem-solving behaviours.

1. Problem. A problem is a task in which the subject is required to attain a goal and the solution cannot be obtained directly by rote memory. In this study, routine problems refers to mathematical tasks that are isomorphic to examples used in the instruction. Other novel tasks are referred to as nonroutine problems.

2. Problem-solving. Problem-solving is the activities of the subject, overt and covert, involved in trying to attain the goal in a problem. (The subject may end up with a wrong answer or in vain.)

3. Problem-solving performance. This refers to all overt behaviours of the subject involved in problem-solving, which include the written solution, thinking-aloud protocols, and any other observable behaviours.

4. Problem-solving process. This is the process by which the subject starts from the given information and tries to attain the goal of the problem.

5. Problem-solving strategy. This refers to any method or algorithm involved in the problem-solving process. It can be a method of extracting relevant information from the given problem, a method of representing the problem in

a diagram or whatever forms, a method of transforming the given information, or a method of search for relations between elements in the problem. The problem-solving process is basically controlled by the application of these strategies. Domain-specific strategies are those applicable in a specific subject area. General strategies are applicable in a wide variety of domains. Means-ends analysis and pattern searching are examples of general strategies.

6. Heuristics. This is a broad class of general problem-solving strategies which are efficient in searching for solution paths. In mathematics, methods discussed by Polya (e.g. 1945) are such heuristics that can be applied in various topics of mathematics.

7. Problem representation. This is the task situation as perceived by the subject and consists of the relationships between the elements in the problem. It may contain errors in the interpretation of the tasks. It may be different from the perfect problem representation of an expert in the subject area. However, it is certainly closely related to the given problem (Simon, 1978). This can only be inferred from the subject's overt behaviours such as written solution, diagrams drawn or thinking-aloud protocols.

8. Subproblem. A subproblem is a problem in itself formulated from a given problem by breaking it into parts. The solution of subproblems is usually easier and can contribute to the solution of the original problem.

9. <u>Subgoal</u>. This refers to the goal of a subproblem.

10. <u>Thinking-aloud protocol</u>. This refers to the verbatim transcript of a cassette recording made of all verbalizations produced by the subject during the problem-solving process. It contains a complete record of the description of the problem-solving process provided by the subject's thinking-aloud activity. It can indicate the sequence in which problem-solving is proceeded.

11. <u>Protocol analysis</u>. This refers to the qualitative and quantitative analyses made on the thinking-aloud protocols transcribed from recordings made in thinking-aloud problem-solving sessions.

12. <u>Knowledge structure</u>. This refers to the subject's organization of knowledge which includes concepts, facts and procedural rules. It can neither be observed directly nor be assessed with complete certainty. It is inferred by the subject's performance in some given tasks in that knowledge domain. We can only build up such <u>inferred</u> knowledge structure which should reflect the knowledge structure possessed by the subject as precise as the sensitivity of the tools designed should allow. A knowledge structure is usually represented as a network showing the various relationships between concepts, facts and rules. The knowledge structure obtained by a rational analysis of the subject matter or possessed by an expert in that domain is referred to as the <u>standard</u> knowledge structure.

13. <u>Conceptual knowledge</u>. This is characterized as the

knowledge that is rich in relationships which link up discrete pieces of information (Hiebert et al., 1986). Thus, a unit of conceptual knowledge cannot be an isolated piece of information and must be a part holding relations with other pieces of information.

14. *Procedural knowledge*. Procedural knowledge in mathematics consists of two kinds of information (Hiebert et al., 1986). One kind is the use of the symbols of the system and the syntactic conventions for acceptable configurations of these symbols. The second kind includes the rules or procedures for solving mathematical tasks. Procedural knowledge is sequential in nature. Several levels of procedures may combine to form a super-procedure.

15. *Structural diagram*. This is a diagram showing the mathematical relationships between different concepts or rules in the subject area or between different components in a mathematical problem. It can illustrate the structure of a subject area or a mathematical problem. (See examples of using structural diagrams in the instructional material shown in Appendix C.)

Hypotheses

In view of the results from expert-novice studies and the hypothesized functions of the structural diagrams, the following hypotheses are formulated:

1. There will be qualitative differences in the inferred knowledge structures between students of

good and poor performance, with regard to the richness of relationships, the organization of levels of facts, and the reference to procedural rules.

2. There will be qualitative differences in the problem-solving processes between students of good and poor performance, with regard to the representation of the problem, the structure of solution, and the type of strategies used.

3. There will be qualitative differences in the inferred knowledge structures between students of the experimental group and the control group, with the knowledge structures acquired by students in the experimental group in general being more close to the standard knowledge structure. This difference will be greater for students in the low ability classifications in the two groups, showing interaction effect of the instructional method and the students' ability level.

4. There will be qualitative differences in the problem-solving processes between students of the experimental group and the control group, with students in the experimental group performing better with regard to the representation of the problem, the structure of solution, and the type of strategies used. This difference will be greater for students in the low ability classifications in the two groups, showing interaction effect of the

instructional method and the students' ability level.

5. There will be significant difference in each of the "routine problem" score, the "nonroutine problem" score, and the total problem-solving score between the two groups with the mathematics achievement test score as a covariate.

6. There will be significant interaction of instructional method and the students' mathematical reasoning ability level on each of the three problem-solving test scores.

<u>Subjects</u>

<u>The pilot study</u>. Twelve students (two Form 6 and ten Form 5 students of which eight were boys and four girls) who had learned the chosen topic already were invited to attend thinking-aloud sessions to solve some problems in that topic. They came from the same school (referred to as School P below) but different classes. In terms of general mathematics performance, four students were chosen from each of the high, middle and low classifications.

Two classes (one science class and one arts class) of Form 5 students were then chosen from School P for a pilot test of a written problem-solving test on the chosen topic. There were totally 74 students (35 boys and 39 girls) who represented a broad range of general mathematics performance. According to the test scores which reflected their performance in this specific topic, five students (one, two and two students from the high, middle and low performance groups respectively) were chosen to attend thinking-loud sessions for a pilot test of a revised battery of items for task-based interviews.

<u>The main study</u>. Two Form 3 classes were involved in the experimental study. The selection was not random and depended primarily on the willingness of the school principal to participate. They were from two Anglo-Chinese subsidized schools (referred to as A and B below) of comparable academic standard and located in the same district. In the class selection process, reference had also been made to students' scores in the mathematics attainment test which was a common test from the Education Department administered at the end of Form 2. Since it was taken about eight months before, the scores served only as a rough indicator, ensuring that their general performance in mathematics would not differ greatly. Both classes contained students of good and of poor performance in mathematics. Both classes were also one of the academically better Form 3 classes in their own schools. The age and sex of these subjects are shown in Table 1.

At the end of the instruction sessions, 6 students were chosen from each class to attend task-based interviews. There were 3 students from the high and the low ability groups respectively for each class.

Although students from separate schools were involved, this would not cause much problem in terms of their mathematics background. Both classes followed

Table 1

Age and Sex of Subjects in the Main Study

	Class of School A (Control Group)	Class of School B (Experimental Group)
Age		
Mean (year)	15.1	15.1
SD (year)	0.73	0.70
Sex		
No. of boys	19	16
No. of girls	19	20
Total	38	36

roughly the same teaching schedule in mathematics during the academic year. Furthermore, the topic chosen for this study was not related to other mathematics topics covered in Form 3. Normally, no Form 3 students would have covered the chosen topic in any formal way.

Materials

1. Mathematics achievement test.

This test was used for the assessment of students' mathematics achievement in the mathematics curriculum. It was adapted from a mathematics test used formerly as a scaling test in the Junior Secondary Education Assessment. The original test was a multiple-choice test consisting of 52 bilingual items, covering a wide range of topics in the junior secondary school mathematics curriculum. As it was designed by the

Education Department as a public scaling test, the test items had high reliability.

With reference to the actual Form 3 teaching schedules of the two participating schools, some obsolete or inapplicable items were deleted. Three experienced teachers commented on the suitability and difficulty level of selected items. The final test adapted contained 40 items and took one hour to complete (see Appendix A).

2. <u>Mathematical reasoning test</u>.

This test was used for the assessment of students' reasoning ability with mathematical quantities, symbols and geometrical figures. It was adapted from a published battery of multiple-choice items specifically designed for Form 3 students (Siu, 1979). The original test had 35 items. Some items were deleted to cut it short. The adapted test contained 30 items and took 40 minutes to complete (see Appendix B).

3. <u>Instructional material</u>.

The topic chosen for instruction was an algebra section on arithmetic progressions which should be in the Form 4 mathematics curriculum. This topic could be considered as quite detached from other topics in the curriculum so that effects of knowledge from other topics on the study would be reduced. It would not be a difficult topic for Form 3 students since the basic ideas involved were sequences of numbers bearing a simple pattern in which general problem-solving strategies could be applied in many cases, even without specific knowledge of the topic. However, this topic would not be taught to the same level of sophistication as in Form 4. More difficult types of problems were deleted.

Based on the standard knowledge structure of this topic and the common mistakes observed in the pilot study, instructional materials were designed. The content and order of presentation of the materials were next reviewed by three experienced teachers. The instructional materials were then revised. For students' use, a printed booklet written in English was also prepared, covering the content and examples, and containing the practice questions. The contents were divided into 4 units to be covered in four one-hour sessions as outlined in Table 2. (For details, see Appendix C.) The booklets for both classes would be almost identical except that structural diagrams were included in the text for the experimental group. (For using structural diagrams in examples and discussions, refer to Appendix C.)

Table 2

An Outline of the Instructional Material

Unit	Content
1	Sequence and Arithmetic Progression * Concepts of sequence and A.P.[1] * First term and common difference of A.P. * Formula for finding the nth term of A.P.
2	Terms in an Arithmetic Progression * Use of the general term formula in problems involving terms and order of terms
3	Summation of an Arithmetic Progression * Derivation of the summation formulas * Use of the summation formulas in simple problems
4	More Examples on Arithmetic Progressions * More worked examples in A.P. using all the formulas covered in Units 1 to 3

[1] A.P. stands for 'arithmetic progression'.

4. Problem-solving test.

This test was used for assessing students' problem-solving performance in the topic instructed. The test was designed with reference to the content covered in the instruction treatment. Items were developed to test the following:

(i) the understanding of the basic concepts and the direct application of formulas in the topic,

(ii) typical problems similar to those demonstrated as examples,

(iii) harder problems requiring the combination of several formulas,

(iv) harder problems requiring clear problem representations in terms of these mathematics concepts before formulas could be correctly applied.

This written test contained two sections. Problems of the types (i) and (ii) formed the section on routine problems. Those of (iii) and (iv) formed the section on nonroutine problems. Separate scores would be given to the two sections, yielding the routine problem score, the nonroutine problem score and the total problem-solving score.

The first batch of items (with 7 routine and 6 nonroutine problems) were reviewed by three experienced teachers to ensure validity. It was then pilot tested, the time allowed being one hour. The test scripts were scored according to a scoring scheme which allocated marks for right solution method, right intermediate answers and promising approaches separately.

Based on the results of the pilot test, some items were revised. This included changes in wording to avoid ambiguity, the order of the problems, and one more simple question to test conceptual understanding. The scoring scheme was also revised, incorporating some unexpected responses. The revised version of the test contained 8 routine problems (5 marks each) and 6 nonroutine problems (8 marks each), the maximum score for each section being 40 and 48 respectively. Notice that the test was bilingual to ensure understanding at the literal level. It was estimated that one hour would be sufficient for the majority of students to complete the test. As this was supposed to be a power test, the time was set to be 1 hour 15 minutes in its administration. (For details of the test, refer to Appendix D which also includes the scoring scheme.)

5. Task-based interview.

Items employed in the task-based interview were to elicit responses that could be used for constructing the knowledge organization of the subject. Problems were written to check the subject's understanding of several basic concepts, mastering of procedural rules, and their readiness in applying the knowledge.

In the pilot study, problems written were tried out on twelve students. According to the performance of the subjects, more problems were written and some were revised. These items were also assessed by experienced teachers in terms of the coverage and the expected responses from subjects of different level of achievement in the topic. Twenty items were then selected for another pilot test on five more subjects. Based on their responses, these 20 items were slightly revised and rearranged and formed the final batch to be used in the actual interviews in the main study. The test items ranged from simple questions, problems involving direct application of formulas to problems involving a combination of procedures. They would be presented to the subject in order of difficulty, with the simplest one first. Again these items were in bilingual version. (See Appendix E for these items.)

Procedures

This research was carried out in two stages, the pilot study followed by the main study.

The pilot study. In this stage, the instruments for the main study were developed and pilot tested. Students who had already learned the chosen topic were interviewed. The knowledge organization from this investigation also provided some of information required for the design of instructional materials to be used in the main study.

Twelve students from Forms 5 and 6 attended task-based interviews using the first trial batch of problems. They had learned this topic at least six months ago. Based on these preliminary data, some items were revised and more items were written. At the same time, a written problem-solving test on this topic was developed and then pilot tested using two classes of Form 5 students. This test was then revised. (The developed written test would be used for assessing Form 3 students in the main study so that Form 5 students were not a very good choice for the pilot test. But, it was impossible to find classes of lower forms who had learned this topic because it was usually taught at the end of the Form 4 mathematics curriculum.)

Using the scores from the pilot problem-solving test, five students of different performance in this topic were chosen to attend task-based interviews using a revised set of items. This time, the emphasis was on the middle and the low performance groups who were expected to possess a knowledge structure not close to the standard one. Their responses would be more useful in making revision on the items. All task-based interview sessions were audiotaped and later transcribed as protocols for analyses.

Results from these studies also contributed to the preparation of the instructional materials. Special attentions were given to some common mistakes observed. Examples were designed to clarify these points. Lessons were planned with a suitable amount of class exercises.

At this stage, three experienced teachers were invited to comment on the validity of the test items, the coverage of the content by these items, and the appropriateness of the instructional materials.

The main study. The second stage was the main study. Two classes of Form 3 students participated in the experiment, one being the treatment group and the other the control. All the meetings were held after the normal school hours. Before the instruction sessions, the mathematics achievement test and the mathematical reasoning test were administered.

Both classes attended four one-hour lessons arranged within one week. For each one-hour session, teaching took about 30 minutes and the other 30 minutes were spent on classwork and explanation of the solution. During the lessons, a printed booklet with text materials and some practice exercises was delivered. The instruction followed the printed booklet closely to ensure uniformity.

Instruction was conducted by the same person in both classes and care had been exercised to ensure equal enthusiasm and style of presentation in both classes. Following their usual practice, the medium of instruction was Cantonese though English text materials were used. Notice that the instruction on worked examples were almost identical. Analysis of problem-solving techniques was given for both classes except that for the experimental group, this analysis was accompanied by structural diagrams. Practice questions were done and checked in class. Further elaboration to students' queries on these exercises had been kept to a minimum to ensure uniformity in materials covered. Instructional booklets were collected at the end of each lesson.

The problem-solving test was administered on the next day immediately after the instruction sessions. Six students were selected from each class to attend task-based interviews which aimed at studying their acquired knowledge structure of the topic. In this case, three students were from each of the low and the high ability groups. These interviews were arranged within one week from the last day of instruction. It was not practical to return for a second interview since it was quite difficult to arrange with the school and the student might forget the details after one to two weeks. So all chosen subjects were only interviewed once. The time taken for the interviews ranged from 1 hour to 1 hour 20 minutes. All task-based interview sessions were audiotaped and then transcribed for subsequ[illegible].

In the experimental group, actually 7 subjects had been interviewed. One of them did not make much verbalization so that the session had to be discarded and one more student was interviewed. Furthermore, owing to infectious disease, 6 students of the experimental group had been absent from some of the test or instruction sessions. Their data were deleted from the statistical analyses. The final sample was such that the control group consisted of 38 students (19 boys, 19 girls) while the treatment group had 30 students (13 boys, 17 girls).

Data Analyses

In the main study, the data collected from the task-based interviews were subjected to qualitative analysis with the following points of attention:

1. What were the concepts understood by the student?
2. What were the procedural rules learned by the student?
3. How were these rules related to the various concepts?
4. What were the common errors in using the procedural rules? What were the underlying misconceptions?
5. What were the problem-solving strategies employed?
6. How were the problems represented by the students?
7. What were the characteristics of the problem-solving process?

Knowledge structures obtained from the two groups of students were constructed and compared in terms of the richness of relationships, relations with procedural rules, levels of organization, and proximity to the standard knowledge structure of the topic. The relationships between the knowledge structure inferred and the problem-solving performance observed were also analysed.

To test the hypotheses on the problem-solving test scores, the following statistical method was employed: two-way analysis of covariance on each of the problem-solving scores, namely the routine problem score, the nonroutine problem score and the total problem-solving score, between the instructional method, and the students' mathematical reasoning ability level with the mathematics achievement test score as a covariate.

Limitations

1. In the main study, random assignment of subjects was not possible and intact groups were used instead. There was the possibility that some uncontrolled factors had influenced the results. This also placed limits on the generalizability of the results obtained to the student population.

2. Acquired expertise is a matter of learning and practices for quite a long time. The instructional treatment in the present study lasted for only 4 hours. The mathematics content covered only amounted to about one week's work in the usual mathematics curriculum and was therefore quite limited. So, we can study only a few consequences of the learning outcomes within a small

subject area. The results may be rather limited in scope in terms of content.

3. One consideration for the choice of the topic (arithmetic progression) is that it can be treated at length without much interaction with other topics in mathematics. (This is similar to the choice of artificial puzzles in early problem-solving studies to eliminate knowledge interaction.) This advantage is its own shortcoming because it would not be a representative topic in the secondary school curriculum. Thus, the results from this study of problem-solving performance may not be generalized much further to other topics in the mathematics curriculum which usually bear rich relationships to one another.

4. A precise model for knowledge acquisition of a learner requires a detailed documentation of the knowledge structure at different stages of learning. The tools employed here are not sensitive enough to assess minor changes and subtle individual differences so that the present study can investigate only the overall effects of the instructional programme.

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CHAPTER V RESULTS AND DISCUSSION

The Pilot Study

Standard Knowledge Structure of the Topic "Arithmetic Progression"

To prepare the test items and instructional material for the main study, the standard knowledge structure of the topic "Arithmetic Progression" was first constructed by a rational analysis of the content. Three experienced mathematics teachers assisted in reviewing the knowledge content. Figure 3 shows the standard knowledge structure, representing the organization of the content as perceived by mathematics teachers. A high-level descriptive representation is adopted here in which the knowledge content and their relationships are emphasized. No claims are made with regard to the precise form of psychological representations of knowledge within the human memory. This is primarily because the present study did not aim at assessing the organization of the long-term memory and the instruments were not designed for that purpose. Secondly, as suggested by Heyworth (1988), the low-level representation which depicts the structure of the human memory may not be very useful in the educational context.

An overview of the standard knowledge structure. In Figure 3, the standard knowledge structure starts with the central notion of an arithmetic progression at the top left corner. Basic concepts such as the "first term" and

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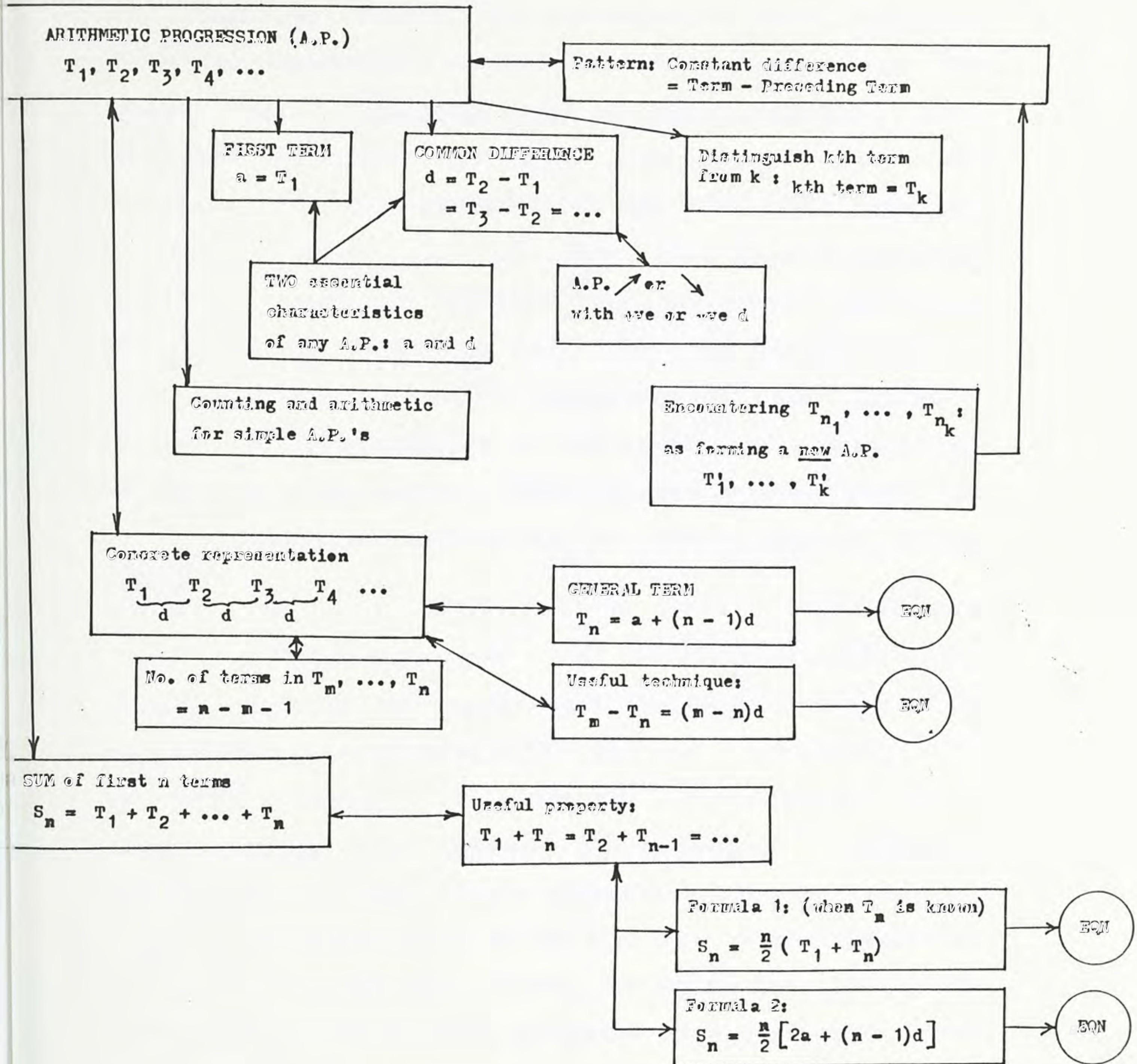


EQN : The formula can be applied to set up equations.

Figure 3. The standard knowledge structure of the topic "Arithmetic Progression".

the "common difference" are presented at lower positions which represent the level of conceptual knowledge. The content shifts from the level of basic concepts to the knowledge of procedural rules down the structure and increases in sophistication down the figure and from left to right. However, it should be noted that conceptual knowledge cannot be isolated from procedural knowledge. For instance, procedural rules have to be executed to distinguish or to identify examplars of these concepts. Arrows (→) are drawn in to represent strong association of one item with another. Double-headed arrows (↔) can be interpreted as indicating two items closely inter-related.

Three major parts can be distinguished in the diagram. Firstly, the upper part begins with the idea of an arithmetic progression (A.P.) as a sequence of a specific pattern of constant difference. Basic terminologies are defined in terms of a symbolic representation. Some simple properties are observed. For example, sub-sequences from an A.P. can be interpreted as a new A.P. if the same pattern is satisfied. The middle part deals with the terms generated from some given terms of an A.P. The formulas and procedures for manipulating terms and the number of terms are based on a more concrete representation of an A.P. which can facilitate understanding of the formulas. The lower part handles the sum of terms in an A.P. Two formulas are associated with the calculation of this sum. The derivation of these formulas rests on a property observed in A.P.'s.

Notice that the existence of some items in Figure 3 demonstrates better understanding of the subject matter. For example, when the idea that

$$T_1 + T_n = T_2 + T_{n-1} = \ldots$$

is related to the summation formulas, these formulas can be mentally derived easily. When the formula for the general term is related to a concrete representation of an arithmetic progression, the formula can be derived by simple counting. Therefore when a student perceives such relationships, it may be inferred that he or she gets a better understanding of these formulas.

'EQN' in the diagram represents the knowledge of using these formulas to write equations when some variables on the right side of the formulas appear as unknowns. Although this is trivial to students of senior forms, junior students may not immediately recognize the utilization of these newly learned formulas as a resource for setting up equations. Alternately, 'EQN' can be interpreted as the knowledge of using the formula reversibly by algebraic operations.

Here, foundation knowledge of elementary algebra is assumed and this does not specifically enter into the knowledge structure shown. For example, recognition of the algebraic symbols and the suffix notation (e.g. T_1) and the substitution of values into formulas are taken for granted. Needless to say, general strategies are also necessary in the execution and the synthesis of these procedures.

Figure 3 represents the knowledge structure for this topic in the secondary school mathematics curriculum only. On the other hand, to a mathematician, some items are not important because they can be automatically deduced within a broader mathematics context. Based on the standard knowledge structure constructed, questions and problems were written to tap students' understanding of these concepts and execution of respective procedures. Instructional materials were written to cover each part of the knowledge structure.

The Written Problem-Solving Test

The written problem-solving test on this topic was pilot-tested using two classes of Form 5 students. The test-scripts were marked according to a scoring scheme, yielding a routine problem score, a nonroutine problem score and their total. Table 3 summarizes the statistics of these test scores. The means of the three scores were very high relative to their maximum scores, showing that these test items were quite easy to Form 5 students. These statistics might also provide a standard to compare the mastery of the content by the Form 3 students in the main study with that by the Form 5 students in a normal course of study.

Table 3

Means and Standard Deviations of the Scores in the Pilot Problem-Solving Test

Test Score [1]		M	SD
Routine Problem Score	(35)	31.19	4.80
Nonroutine Problem Score	(48)	33.15	11.78
Total Problem-Solving Score	(83)	64.34	15.00

No. of subjects = 74

[1] The maximum score for each measure is shown in brackets.

Reliability of the test scores. To determine the internal consistency of the test items, the coefficient alpha was computed. Table 4 presents the reliability coefficients of the three sets of test scores.

The reliability of the routine problem score was slightly low. This was probably because the items were designed to assess the learning outcome of Form 3 students later and thus appeared to be relatively easy for Form 5 students. This reason was supported by the high mean and small standard deviation for the routine problem score in Table 3. Therefore, a higher reliability coefficient for this score could be expected in the main study.

Analyses of the written responses. Apart from the necessity to revise some of the items based on the written responses collected in the pilot test, errors and mistakes

Table 4

Reliability Coefficients of Problem-Solving Test Scores in the Pilot Test

N = 74

Test Score	No. of items	Alpha
Routine Problem Score	7	.73
Nonroutine Problem Score	6	.78
Total Problem-Solving Score	13	.81

were identified. Errors could occur at different stages of the solution process: (1) understanding and representing the problem, (2) retrieving or searching for relevant procedures, and (3) executing the procedure. However, misunderstanding a problem or an inappropriate problem representation could not be discovered unless it came to be solved by some procedures and be written down by the subject. This laid down some limitations to the present analyses.

Arithmetic or algebraic slips occurred very often in the execution stage. These included mistakes in transposing terms, simplifying algebraic expressions, substituting values for variables, etc. But they were not pertinent to the present study. Instead, special attention was drawn to wrong procedures that were systematically applied to solve problems. Table 5 summarizes the common mistakes observed in the written solution. These were found to be quite typical in the wrong solution of problems by many subjects. They were also further confirmed by data collected in pilot task-based interviews.

Table 5

Common Mistakes in the Topic "Arithmetic Progression" as Observed in the Pilot Problem-Solving Test

Knowledge item	Common mistakes observed
Arithmetic progression (A.P.)	• include other sequences (non-A.P.'s) in calculation without checking the condition for A.P.'s
First term	• keep it as a fixed constant without altering its value when sub-sequences* are considered
Common difference	• use $d = T_1 - T_2$ when $T_1 > T_2$ (in this case, d should be negative) • keep it as a fixed constant without altering its value when sub-sequences are considered • take $T_n - T_1 = nd$ or equivalently $d = \frac{T_n - T_1}{n}$
Number of terms	• take T_m, T_{m+1}, ..., T_n as containing (n-m) terms • consider that (n-m) terms lie between T_m and T_n
General term	• wrong formula: $T_n = a + nd$
Sum	• wrong formula: various

* A sub-sequence is a new sequence formed by terms selected from a given sequence.

With reference to the standard knowledge structure shown in Figure 3, these common mistakes can be interpreted as follows. The inclusion of other sequences that were not A.P.'s into A.P. calculation suggests that some students did not examine the pattern of constant difference <u>formally</u> for <u>all</u> the terms particularly when some irregularities in the sequence were obscured by the surface information. Problem 13 in the pilot test (i.e. Problem 14 in the Problem-Solving Test for the main study) is an example where students applied the formulas for A.P.'s without noticing that the quantities did not form an A.P. unless some terms were deleted. (See Figure 4 for details.) Perhaps, students were not sensitive enough to pick out such irregularities or they had been too quick to generalize patterns from a few terms in the sequence. This was supported by the fact that most often, only the beginning terms were observed in their written solution.

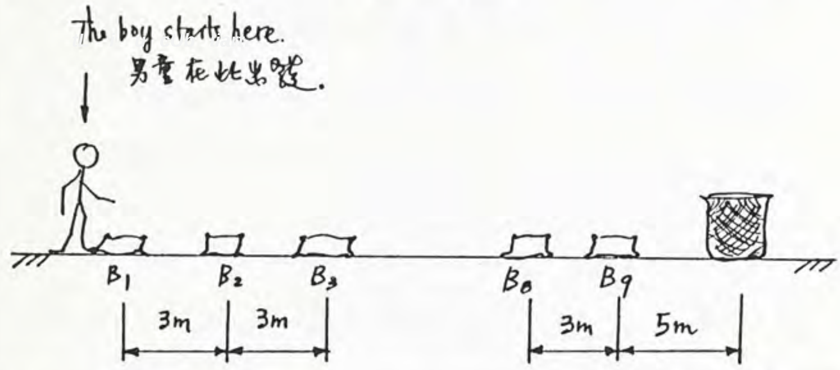
When sub-sequences of a given A.P. were encountered, some students failed to recognize that the parameters for the new A.P.'s <u>may</u> be different from the original one. This occurred if they did not re-order the sub-sequence as an A.P. in its own right, i.e. giving new suffixes to the terms:

$$T_{n_1}, T_{n_2}, \ldots, T_{n_k} \text{ as } T_1', T_2', \ldots, T_k'.$$

For decreasing A.P.'s, the common difference d was sometimes mistaken to be positive. This reflects an ignorance of the formal definition of d which actually implies a sign convention such that increasing A.P.'s have positive d's and decreasing A.P.'s negative d's.

In playing a game of picking up sand-bags, a boy is asked to pick up 9 and-bags and put them into a basket, one at a time. The bags are arranged in a straight line and the distance between two consecutive sand-bags is 3 m (see figure). The basket is placed 5 m away from the last sand-bag B_9. The bay starts at B_1, pick up sand-bag B_1 and runs to the basket. Then he returns to pick up B_2 and run to put it in the basket again. This repeats until he puts the last one (i.e. B_9) into the basket. Find the total distance travelled by the boy in this game.

在一拾沙袋的遊戲中，男童須要逐一拾起沙袋共9個，放進一竹籃。沙袋排成一直綫，相鄰沙袋各相距 3m (見圖)。竹籃與最後沙袋 B_9 相距 5m。男童從 B_1 出發，先拾起 B_1，走到竹籃處放下。接着返回 B_2，拾起沙袋 B_2，再奔回竹籃，如此類推，直至將最後一個沙袋(即 B_9)放進竹籃為止。求男童在遊戲中共跑了多少路程。



[This is Problem 14 in the Problem-Solving Test for the main study.]

The distance travelled (in metres) by the boy can be represented by any one of the following sequences:

(i) <u>29</u>, 26, 26, 23, 23, 20, 20, ..., 8, 8, 5, 5 (17 terms)

(ii) <u>29</u>, 52, 46, 40, ..., 16, 10 (9 terms)

(iii) 55, 49, 43, 37, ..., 19, 13, <u>5</u> (9 terms)

Which one is chosen depends on how one breaks down the segments travelled. But in any one of these, there is a single term (underlined above) which must be taken apart in order that the remaining sequence can be an A.P. (in (i), two identical A.P.'s).

<u>Figure 4</u>. Problem 13 in the pilot problem-solving test.

While there were only a few students putting down the wrong formula $T_n = a + nd$ to evaluate the term T_n in their solution, about ten students had used the wrong relationships $T_n - T_1 = nd$ or $d = \frac{T_n - T_1}{n}$ to calculate the common difference d. It is interesting to note that all the three formulas are equivalent algebraically. But students more easily committed the error of using the latter two relations. This may be attributed to the usual slip in counting in which the number of segments between n points in a straight line is mistaken to be n instead of (n - 1). The same reason explains why the number of terms in or between T_m, ... , T_n was so often taken wrongly too.

If the procedures shown in Figure 3 for finding the number of terms and the difference $T_m - T_n$ were formally executed in such circumstances, these mistakes would be avoided. Of course, the understanding behind the procedure is also emphasized in the standard knowledge structure. However, these two procedural rules do not appear in any current textbooks and they are seldom explicitly used by the teachers in their instruction. It is generally assumed that students can get these quantities correct by simple counting or subtraction. In fact, as the written responses in the pilot test showed, these simple procedures were not systematically employed by the students in their calculations.

The summation formulas are slightly complicated compared with other algebraic rules. So, wrong formulas for summation appeared more often than the general term formula. Various expressions were observed and were probably arised from a hazy memory of the variables involved. To resolve this difficulty, in the instruction material for the main study, it is suggested that these summation formulas should be related to a property of the sum of A.P.'s, namely that

$$T_1 + T_n = T_2 + T_{n-1} = \dots ,$$

so that the formulas can be readily understood in terms of this property as shown in the standard knowledge structure. It is supposed that when meaning is attached to the terms in these expressions, the formulas can be understood and better memorized.

Furthermore, in this topic of algebra, problems can very often be solved with general algebraic techniques instead of using specific knowledge of arithmetic progressions. For example, consider Problem 6 in the pilot problem-solving test (which is Problem 5 in Appendix D):

> The 4th term of an A.P. is 11.
> Its 7th term is 35.
> Find its 18th term.

Most students in the pilot test solved it by setting up two algebraic equations:

$$\begin{cases} a + 3d = 11 \\ a + 6d = 35 \end{cases}$$

After solving a and d ($a = -13$, $d = 8$), they found the 18th term by calculating $a + 17d = 123$. This solution is excellent algebraically but requires only the knowledge of the general term formula to express the nth term.

By contrast, with an elaborated representation of the A.P. as

$$T_4 -- T_5 -- T_6 -- T_7 -- T_8 -- \ldots -- T_{18} ,$$

where a stroke (--) stands for one common difference d, we can immediately get by simple arithmetic,

$$d = (T_7 - T_4) \div 3 = 8$$

and $$T_{18} = T_7 + 11 d = 35 + 88 = 123.$$

This second approach certainly reflects a more flexible use of the properties of A.P.'s and includes the general term formula as a special case (i.e. $T_n = T_1 + (n - 1)d$ in particular).

Briefly to say, effective utilization of knowledge in the topic can simplify the solution procedure as shown above. Judging from the written solution, we may say that the subjects in the pilot test tended to rely on their general algebraic knowledge. This is quite probable because these Form 5 students learned the topic over six months ago and they should be skilful enough to solve the problems by algebraic methods. Yet, there is still a possibility that after learning the topic, some students did not develop an effective representation of the arithmetic progression to assist their problem-solving and formulas remained abstract on a symbolic level. They might solve problems very well on an algebraic level but might not get a more intuitive understanding of the elements in the problems. To mathematicians, more intuition is better than a purely algebraic representation.

In order to provide effective instruction on the topic to all subjects in the main study, these commonly observed mistakes were clearly brought to students' attention in the instruction. A concrete representation of the A.P. as used above was employed to encourage intuitive counting and checking (see Appendix C). Before execution of the formulas, the first term and common difference of the particular A.P. considered were explicitly identified. These measures ensured that all subjects would receive a meaningful exposition of the material before any comparison of their learning outcome was to be made.

From an alternative perspective, results depicted in Table 5 also show that students could hold to wrong procedures and formulas even when these were obviously inconsistent with simple facts. For example, the formula $T_n = a + nd$ which was used by some students could be immediately rejected either by substituting $n = 1$ or 2 or by referring back to the meaning of an arithmetic progression. (To almost all students, it should be obvious that $T_1 = a$ and $T_2 = a + d$.) Procedures in obtaining the number of terms could be verified or rejected by direct counting. However, students were observed to have applied these procedures repeatedly without noticing the absurdity of their rules.

There may be several reasons. Students might just handle these formulas only by rote memory. Their knowledge lacked understanding so that pieces of information were isolated from other mathematical knowledge instead of integrated into a coherent network. As reviewed earlier, conceptual knowledge is characterized by the richness in relationships. When students did not actively relate pieces of information or they were not encouraged to do so, this state of poor understanding would continue and wrongly memorized formulas would not be doubted.

Pilot Task-based Interviews

With the task-based interviews based on the first trial batch of items, it was found that some kinds of questions were not appropriate for the purpose of assessing knowledge structures. Apart from providing hints for the revision of items, the protocols collected revealed two points of interest.

Firstly, three subjects from the high ability group (in terms of general mathematics performance) solved simple problems smoothly, but with two diverse approaches. On reading the statements of the problem, two subjects just built up correspondence between the problem elements and the formal symbolic representation of the subject matter. They mainly recalled the relevant formula immediately upon identifying the essential variables and directly substituted in the formula to find the answer. Another subject performed differently. In some cases, he repeated the reasoning procedure in the derivation of formulas using the givens and unknowns of the problems. Reasoning procedures and understanding were exhibited in his verbalization. The following problem is an example.

> The numbers 6, x, (33-x), ... form an A.P.
> (a) Find the value of x.
> (b) Find the 35th term.
> (c) Find the difference between the 10th term and the 20th term.

After correctly finding x to be 13 in (a), he continued:

> x is 13 so that the common difference is 7.
> The 2nd term adds one 7. ... The 35th term
> adds thirty-four 7. Therefore, 6 + 34 x 7 ...
> [writing down and using the calculator]
> 244 is the answer.

The underlined statement would not appear if the subject did not go through the reasoning step again. He could actually write down the formulas correctly in other problems. When he was asked about why he did not apply these formulas in such cases, he gave the following reply:

> I would be more sure of the problem. ...
> The formula can be easily remembered if
> I do it this way.

This use of general mathematical strategies can be considered as a kind of self-justifying steps in executing familiar procedures. It will make such an attempt to solve problems a renewing process of learning and understanding too. (See later discussions on problem representations in the main study.)

Secondly, the use of backward reasoning was seldom observed among the interviewees. It was only used by some more able students in a few problems. In less routine problems, subjects of better performance very often worked forward from the givens by breaking down the problems into stages, each of which was a familiar task. Subjects of poor performance also proceeded from the givens but with less well-defined aims. They might be described as making some random search, substituting the givens into any formula they would think of. This process to generate more variables or expressions was continued in the hope of reaching the final goal.

Two reasons can be proposed. On the one hand, the problems in this topic involve a limited number of variables. Students may learn from practice that when they work out a sufficient number of parameters, a problem can normally be resolved. Thus, backward reasoning from the goal is not necessary. On the other hand, successful backward reasoning requires a clear formulation of the goal and givens, an understanding of the relationships existing between these variables as well as logical reasoning steps. Less able students may simply not be capable of working with this strategy. For more able students, they will perceive a problem in terms of familiar subproblems rather than a network of relationships between variables. This tendency may be explained by the activation of typical problem schemata during problem solving. This observation further supports the suggested design of the present study which was devised to facilitate schema acquisition in mathematics instruction.

With a revised batch of test items for interviews, the performance was found to provide sufficient data for probing the subjects' knowledge organization. In this second trial, the focus was on students of poor performance to check whether the questions would be too difficult for them. Generally speaking, problem-solving behaviours and understanding of the topic observed in the pilot task-based interviews were much liked those in the main study which will be discussed in a later section.

The Main Study

Statistical Analyses of Test Scores

Reliability and descriptive statistics of test scores. For each subject, two test scores were obtained before the treatment programme. These were mathematics achievement test score (MAT) and the mathematics reasoning test score (MRT). After instruction, a problem-solving test was administered and three sets of scores were obtained, namely, the routine problem score (RPS), the nonroutine problem score (NPS), and the total problem-solving score (TPS). The reliability coefficients for these tests are presented in Table 6. Descriptive statistics of these measures are shown in Table 7.

Reliability coefficients for the three scores MAT, RPS and TPS were acceptable. The other two were below .75. For the MRT score, students might be unfamiliar with this type of reasoning test so that the time allowed (40 minutes) was not sufficient for completing the test. For the NPS score, the pilot test using Form 5 students yielded a coefficient alpha .78, higher than the value obtained here. This indicates that these nonroutine problems might still be quite difficult to the subjects from Form 3 classes in the main study after receiving instruction in the topic. This is clearly supported by the low mean values for NPS in Table 7 as compared to 33.15 in the pilot test (see Table 3).

Table 6

Reliability Coefficients of the Written Tests in the Main Study

N = 68

Test Score	No. of items	Alpha
MAT	40	.79
MRT	30	.74
RPS	8	.82
NPS	6	.72
TPS	14	.88

Note. TPS = RPS + NPS

Table 7

Means and Standard Deviations of the Written Test Scores in the Main Study

	Control Group (N = 38)		Experimental Group (N = 30)	
Test score [1]	M	SD	M	SD
MAT (40)	20.79	5.78	24.43	6.11
MRT (30)	14.92	4.51	16.60	4.93
RPS (40)	23.61	9.99	32.70	5.28
NPS (48)	15.76	10.06	22.87	7.51
TPS (88)	39.37	19.17	55.57	11.81

[1] The maximum score for each test score is shown in brackets.

Furthermore, one more item had been added to the routine problem section and the mean score for this item was 4.38. Taking this fact into account, we can see that the mean value of RPS for the experimental group when deducting this item value (i.e. 32.70 - 4.38 = 28.32) was still comparable to that in the pilot test (i.e. 31.19). This observation confirms that subjects in the main study could attain a level of achievement comparable to those in a normal course of study at least in the area of routine problems even though they were two grades junior.

Correlations between test scores. As the measure MAT would be entered as a covariate in subsequent analyses of covariance on the problem-solving test scores, Pearson product-moment correlations were computed among the test scores and are reported in Table 8 for reference. All correlation coefficients were found to be substantial.

Table 8
Pearson Correlations among Test Scores

Test Score	1	2	3	4	5
1. MAT					
2. MRT	.60				
3. RPS	.56	.54			
4. NPS	.70	.60	.82		
5. TPS	.66	.60	.95	.96	

Note. All coefficients are significant ($p < .001$).

Comparison of problem-solving test scores for the two groups. Two-way analysis of covariance (2-way ANCOVA) was employed to determine if there were group difference and interaction effect on each of the problem-solving test scores RPS, NPS and TPS, with instructional treatment and mathematical reasoning ability level as between-subjects variables and with MAT as a covariate. Mathematical reasoning ability level (RL) was defined in terms of the MRT score.

Table 9 reports the means and standard deviations of the test scores in each of six cells in the 2-way ANCOVA'S. Table 10 tabulates the results of 2-way ANCOVA'S conducted on each of the problem-solving test scores.

Table 9
Means and Standard Deviations of the Problem-Solving Test Scores in Each Cell in the ANCOVA

Reasoning Ability Level		Control Group (N = 38)			Experimental Group (N = 30)		
		n	M	SD	n	M	SD
RPS	High	5	32.00	8.12	9	36.89	2.64
	Middle	25	25.20	8.45	16	31.44	4.53
	Low	8	13.38	7.61	5	29.20	6.34
NPS	High	5	27.50	11.35	9	28.44	3.77
	Middle	25	16.42	8.38	16	21.94	6.68
	Low	8	6.38	5.19	5	15.80	7.83
TPS	High	5	59.50	18.41	9	65.33	4.67
	Middle	25	41.62	15.88	16	53.38	9.94
	Low	8	19.75	11.[illegible]8	5	45.00	13.55

Table 10

ANCOVA Results on Problem-Solving Test Scores Classified by Instructional Method and Reasoning Ability Level, Controlling for Mathematics Achievement Test Score

Source of variation	RPS			NPS			TPS		
	df	MS	F	df	MS	F	df	MS	F
Instructional Method (IM)	1	658.275	14.09**	1	176.040	4.02	1	1515.148	10.03*
Reasoning Ability Level (RL)	2	214.968	4.60	2	186.804	4.27	2	800.469	5.30*
IM x RL	2	82.022	1.76	2	18.310	0.42	2	143.821	0.95
Within	61	46.712		61	43.766		61	151.048	

Note. ** $p < .001$
* $p < .01$

Subjects were divided into three reasoning ability levels on the basis of the mathematical reasoning test score. The high and the low levels comprised the top 20% and the bottom 20% respectively.

As Table 10 shows, significant treatment effects were found for the routine problem score (RPS) and the total problem-solving score (TPS). On the other hand, the difference in the nonroutine problem score (NPS) between the two treatment groups did not reach the .01 significance level. This indicates that instruction with the use of structural diagrams led to better performance in solving routine problems and in the overall test performance. But this instruction did not improve the performance in solving nonroutine problems, contrary to the hypothesis made.

The findings imply that the use of structural diagrams facilitated the acquisition of procedures in solving typical problems, presumably through an explicit representation of the problem components and their relationships. The overall test score (i.e. TPS) was significantly raised probably because the routine problem score occupied a high proportion in the total problem-solving score as shown in Table 7 and students could only attain a relatively low score in nonroutine problems. In other words, when students were assessed mainly on solving routine mathematics problems, instruction using structural diagrams certainly helped to improve the performance.

For the nonroutine problem score, no significant effect was observed, there are two possibilities. Firstly, these nonroutine problems might in fact demand higher ability in setting a problem representation within the context of arithmetic progressions before familiar procedures could be applied.

So, even though one group might be more competent with the standard procedures in the topic, both groups showed poor performance in these nonroutine problems because they equally lacked the general ability in analysing the situation and representing the problem with familiar symbols correctly. That is to say, the use of structural diagrams in the instruction had no specific effect on general problem-solving ability, particularly the understanding and representation of a novel problem in proper form. A second possibility is that the duration of instruction was not long enough to improve the performance in general problem-solving strategies. Perhaps, prolonged use of the method in illustrating problem structures can achieve some improvement at least in emphasizing the stage of understanding a problem before searching for methods of solution. (In later protocol analyses, this stage of understanding was actually found to be largely neglected by subjects of the poor performance group in their problem-solving process.)

When the classification of subjects in terms of mathematical reasoning ability was considered, ANCOVA results in Table 10 showed that the differences in the routine problem score (RPS) and in the nonroutine problem score (NPS) between the three ability levels did not reach the .01 level of significance. This means that the performance in these two test scores between the three ability levels did not vary greatly. Significant difference was only found for the total problem-solving score (TPS) between the three ability levels. When a one-way ANCOVA was further conducted on the score TPS with subjects classified into the three reasoning ability levels, significant differences were found, $F(2,64) = 4.65$, $MS_e = 172.1$, $p < .05$. Post hoc comparisons using the Duncan test revealed that the difference between the high and the middle ability groups was not significant and the difference between the low ability group and any other group reached the .05 level of significance.

These results suggested that the three ability groups did not differ much in performance when the RPS and NPS scores were separately analysed. This lack of significant difference may be attributed to the fact that while the mean RPS and NPS scores for these groups followed the trend of the ability levels, the performance of an individual did not depend so much on the general reasoning ability as the actual learning in the specific topic from the instruction programme. This led to high variance in performance within these groups so that the differences did not reach the significance level. In other words, the learning outcome in each of the two aspects, namely routine problems and nonroutine problems, did not bear a substantially high relation with the subject's mathematical reasoning ability.

When the total problem-solving score (TPS) was considered, the statistical analyses indicated that the low reasoning ability group showed significantly poorer performance. This was probably because the TPS score was the sum of RPS and NPS so that the between-group variance had been magnified while the within-group variance remained unchanged and this had brought the group difference to a significant level in the statistical sense. So, the overall learning outcome of the low reasoning ability group appeared to be substantially lower when compared with the higher groups.

Lastly, ANCOVA results also indicated no significant two-way interaction effects of the instructional method and the ability level on all the three problem-solving test scores, contrary to the hypothesis. It is probable that the treatment in the instructional programme was not substantial enough to provoke the higher thinking strategies in the high ability students. Neither was this treatment long enough to bring special advantage to the low ability students who in general needed more time to practise the basic skills.

Another possible reason to explain the findings comes from the analyses of problem-solving protocols to be discussed in detail in the next section. It was observed that there were differences in the use of problem-solving strategies. In the higher ability group, subjects in the treatment group could recognize subproblems more readily in solving less simple routine problems. They just worked forward with clear subgoals. By contrast, subjects in the control group had to work on and think along in order to reach the final goal. That is to say, subjects in the high ability group made improvement in recognizing more typical problem structures through the treatment. On the other hand, for the low ability division, subjects in the treatment group used less random attempts and they could try to aim at some subgoals. In other words, both ability groups were benefited to some extent from the treatment, though in different respects so that no significant interaction effect was observed. If the duration of instruction could be lengthened to some extent, the hypothesized interaction effect might be observed when more able subjects in the control group could also learn the basic problems more thoroughly without the help of structural diagrams while less able subjects in the treatment group could get the maximum benefit from the instruction.

Analysis of Written Responses in the Problem-Solving Test

Apart from the statistical analyses of the problem-solving test scores reported in the last section, a detailed analysis was conducted on the written solution of the test. This would be expected to provide further information concerning the usual methods of solution, the common mistakes, and other special features that might reveal the subjects' utilization of knowledge in the test. Although detailed analyses were also carried out for the protocols collected in task-based interviews, the number of subjects involved would be highly limited, a little less than one-fifth of all participating subjects. So, this qualitative analysis of solutions in the written test would provide a better picture of the general performance of the whole sample and give further hints to understand the group differences revealed by previous statistical methods.

The analysis included frequency counts for the use of formulas or specific techniques or the occurrence of mistakes. Briefly to say, the common errors and mistakes were very similar to those already documented in the pilot test (see Table 5). Subjects in the main study committed more arithmetic and algebraic mistakes presumably because they were two grades behind the Form 5 students in the pilot test. However, their performance in some other aspects was different from that of the pilot study.

<u>Use of standard formulas</u>. Firstly, the three formulas introduced in the instruction were checked up in the test scripts to see whether subjects had applied them or had written them in the usual algebraic form. The results are presented in Table 11. (Results for individual problems are tabulated in Table 12.) Although it is not fair to compare directly these frequency counts between the two groups because the groups were not exactly equivalent in terms of mathematical achievement or mathematical reasoning ability (see Table 7), some general trends are still obvious. For simplicity, the control and the experimental groups are referred to as group A and group B respectively in the tables and subsequent discussions.

It is observed that subjects in group B could memorize and apply the formulas better than those of group A. Confused expressions for the general term formula were almost absent from group B. More subjects in group B could master both formulas for summation. There were about half of the subjects altogether who could only use one

Table 11

Use of Formulas by Subjects in the Problem-Solving Test of the Main Study

General Term Formula

	Group A	Group B	Total
$T_n = a + (n-1)d$ [correct]	26	29	55
$T_n = a + nd$	4	1	5
$T_n = 2a + (n-1)d$	2	0	2
$T_n = a(n-1)d$	2	0	2
no formula observed	4	0	4
	38	30	68

Summation Formula

	Group A	Group B	Total
both formula used			
both correct	12	17	29
wrong formula(s)	0	0	0
$S_n = \frac{n}{2}(T_1 + T_n)$ only			
correct	5	5	10
wrong	0	0	0
$S_n = \frac{n}{2}[\ 2a + (n-1)d\ ]$ only			
correct	10	7	17
wrong	1	1	2
both formula not observed	9	0	9
use explicitly the property $T_1 + T_n$ = constant to sum	1	0	1
	38	30	68

summation formula. It is quite interesting to notice that one subject in group A could actually perform summation by directly using the property of summing the first and the last terms.

As far as the use of formulas is concerned, subjects in the main study worked with more counting so that the wrong formula $T_n = a + nd$ appeared less often. In fact, they tended to use the concrete representation employed in the instruction in solving problems. For instance, to the same problem discussed previously (Problem 6 in the pilot test/Problem 5 in the main study), nearly half of the subjects in the main study worked with $T_7 - T_4 = 3d$ and got it right rather than set up simultaneous linear equations and solved them wrongly (see Table 12). This implies that the concrete representation introduced was more useful for students to manipulate in understanding elementary problems. It follows that they could count the number of terms more accurately in most cases.

Hazy recollection of formulas for summation was again observed and was more difficult to tackle. Of course, summation formulas are a bit complicated. Perhaps, this indicated insufficient practice might be a weakness of the instruction programme. Although attempts had been made in the instruction to relate the formulas to some simple properties of arithmetic progressions, students might not have grasped it as a clue to memorization of these formulas. Only one student attempted to sum the given A.P.'s by using the property explicitly when he just could not remember the formulas. (Failure to use the property to try to sum A.P.'s was also observed in task-based interviews.)

Furthermore, about half of them knew only one summation formula. It is quite interesting to note that more of them tended to remember the more complicated formula among the two (see also statistics for Problems 6 and 9 in Table 12). A possible reason is that it is the more general one. But the instruction aimed at doing the opposite, starting from the easier formula so that the complicated one could be derived at will (refer to the standard knowledge structure in Figure 3). This statistics implies that the teaching was not too successful in this respect. The students could not associate the formulas in the prescribed way.

Other features relating to problem-solving strategies observed in the written solution. Next, several features pertinent to problem-solving methods were also observed in the written solutions. It should be noted that each problem required some procedures specific to it which might not reappear in another problem. So the focus of this analysis varied among these problems. A list of features observed in each problem is tabulated in Table 12 for reference. (For details of the problems discussed, refer to Appendix D.) Generally speaking, in the routine problems (Problems 1 to 8), subjects of group B made less mistakes and were more capable to apply the appropriate procedures. In the nonroutine problems (Problems 9 to 14), the difference in performance between the two groups became less noticeable.

Table 12
Analysis of Written Responses in the Problem-Solving Test

Group A (Control Group) 38 test scripts
Group B (Experimental Group) 30 test scripts

Problem 1

Given sequence	No. of correct responses Group A	Group B	Total
(a) sequence alternating in sign, not A.P.	33	29	62
(b) decreasing A.P., all negative numbers	36	28	64
(c) increasing A.P., all positive numbers	38	30	68
(d) increasing sequence (all +ve), not A.P.	28	23	51
(e) algebraic expressions in A.P.	24	29	53

Problem 2

Mistakes	Group A	Group B	Total
Use $T_n = a + nd$	4	1	5
Use the general term formula in other wrong form such as $T_n = a(n - 1)d$	4	0	4
Confuse the term with the order of the term: Put $T_{10} = 10$.	3	1	4

Problem 3

	Group A	Group B	Total
Correct procedure			
Use $\frac{T_n - T_1}{d}$ to get $n - 1$	0	3	3
Direct counting by listing the numbers	9	0	9
* Most subjects set up an equation to solve n.			
Mistakes			
Take $\frac{T_n - T_1}{d}$ as n	2	4	6
Set up equations by wrong formulas of T_n	1	0	1
Attempt to set up equations by summation formula	3	0	3
Misunderstand the problem to find T_{22}	2	1	3

Problem 4

	Group A	Group B	Total
Correct procedure			
Direct listing of numbers to find T_{21}	2	0	2
Mistakes			
Use $T_n = a + nd$ to find T_{21}	1	0	1
Use the formula for S_n to find T_{21}	2	0	0
Use other wrong formula to find T_{21}	3	0	0
Use the proportion $\frac{T_1}{T_{21}} = \frac{1}{21}$ to find T_{21}	0	2	2

Problem 5

	Group A	Group B	Total
Correct procedure			
Use simultaneous equations	14	13	27
Use $\frac{T_m - T_n}{m - n}$ to find d	7	17	24
By listing numbers and trial-and-error	1	0	1
Use transformation $T'_1 = T_4$ and $T'_4 = T_7$	0	1	1

Problem 6

	Group A	Group B	Total
Use $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ (correct formula)	16	21	37
Use $S_n = \frac{n}{2}(T_1 + T_n)$ (correct formula) (find T_n first)	4	8	12
Summation by direct addition	3	0	3
Find T_n instead of S_n (wrong)	3	1	4
Wrongly take d as +ve (should be -ve)	2	7	9
No attempt	8	0	8

Problem 7

	Group A	Group B	Total
Find two terms first and then subtract (correct procedure)	19	23	42
Find the difference directly as a multiple of the common difference (elegant method)	1	0	1
Find the terms by direct listing of numbers	1	0	1
Find the difference between sums instead (wrong interpretation of the problem)	1	2	3
No attempt	11	3	14

Problem 8

	Group A	Group B	Total
Set up algebraic equations in terms of the first term (correct procedure)	16	19	35
Use the difference between terms to find both terms (elegant method based on the property of A.P.'s)	1	1	2
No attempt	14	4	18

Problem 9

	Group A	Group B	Total
Get the correct number of multiples	22	24	46
Get the last multiple correct	15	21	36
Use $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$	16	14	30
Use $S_n = \frac{n}{2}(T_1 + T_n)$	8	14	22
Summation by direct addition	6	0	6
No attempt	3	0	3

Problem 10

	Group A	Group B	Total
Set up equation using $S_n = \frac{n}{2}(T_1 + T_n)$ (correct approach)	18	21	39
Set up equation using another summation formula (difficult to solve for Form 3 students)	4	5	9
Use $\frac{T_n - T_1}{n - 1}$ to find d (quick method, c.f. Problem 5)	1	2	3
Use a wrong formula for summation	3	0	3
No attempt	9	1	10

Problem 11

	Group A	Group B	Total
Treat the specified sequence as a new A.P.	12	19	31
New A.P. with wrong first term	3	2	5
New A.P. with wrong common difference	6	7	13
New A.P. with wrong number of terms	6	5	11
Use $S_n = \frac{n}{2}(T_1 + T_n)$ to evaluate sum	3	9	12
Use $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ to evaluate sum	9	9	18
Summation by direct addition	16	6	22
No attempt	5	1	6

Problem 12

	Group A	Group B	Total
Add up two givens to get the sum of ten terms (correct; equations simpler)	2	11	13
Consider the difference between the two sums $S_{10} - S_4$ using algebraic expressions (correct; but equations complicated)	2	0	2
Consider the sum of the next 6 terms as the sum of a new A.P. (correct; elegant and general method)	0	2	2
Get the sum of the next 6 terms by direct addition of algebraic expressions (correct; clumsy)	1	0	1
Confuse the given sums as terms (wrong)	7	7	14
Use the summation formula for the sum of the first 6 terms to express the sum of the next 6 terms (wrong; mistake the condition of application of the formula)	5	3	8
Mistake the sum of the next 6 terms as the sum of the first ten terms (probably misinterpret the problem statement)	0	2	2
No attempt	12	1	13

Problem 13

	Group A	Group B	Total
Correctly distinguish two A.P.'s	6	15	21
Cannot distiguish the two A.P.'s	10	13	23
First terms of both A.P.'s correct	4	15	19
Common diff. of both A.P.'s correct	4	11	15
No. of terms of both A.P.'s correct	3	7	10
Get the sum by direct addition	4	0	4
Wrongly consider terms, not sums	8	5	13
No attempt	10	1	11

Problem 14

	Group A	Group B	Total
Represent the lengths travelled as a sequence of terms correctly	2	2	4
Represent the lengths travelled as a sequence of terms with some critical terms wrong	11	13	24
Mistake the whole sequence of lengths as an A.P.	9	11	20
Sum the lengths by direct addition	12	7	19
Find the sum by some wrong multiplication	2	2	4
No attempt	12	7	19

The concrete representation introduced was employed by over half of the subjects in their solution to various problems. This was evidenced by their A.P.'s being written in the form

$$T_1 \text{ -- } T_2 \text{ -- } T_3 \text{ -- } T_4 \text{ -- } T_5 \text{ -- etc.}$$

or by the division $(T_m - T_n) \div d = m-n-1$ directly done without resorting to the general term formula. But they did not tend to use it in all problems. For example, in Problem 5, nearly half of the subjects used it. In Problems 3 and 7, more made use of the general term formula. Compared with the written responses in the pilot test, subjects in the main study used general algebraic techniques less frequently. This observation demonstrates that the representation introduced would be an effective tool in problem-solving.

The results also indicate that there were some students who attempted to solve problems by simply listing the sequence and/or summation through direct addition. In some cases, they understood the problem but did not have the formula available for computation or setting up equations. This happened exclusively in group A (see e.g., Problems 3, 6, 7 and 9 in Table 12). In other cases, mainly the nonroutine problems, subjects might not know how to relate the problem situation to an A.P. and they would resort to arithmetic. Besides those subjects belonging to group A, a few subjects in group B were also observed in this category (see e.g., Problems 11 and 14 in Table 12). This observation supports further that subjects in group B were more competent with the use of formulas but for nonroutine problems which required more skill in problem representation, the treatment using structural diagrams had limited effects.

It is clearly observed that in harder problems (e.g. Problems 13 and 14), students in both groups found difficulty in relating the givens to the A.P. formulation. Compared with Form 5 subjects in the pilot test, they made more mistakes in assuming sequences to be A.P.'s without formally checking the conditions and in keeping parameters unchanged for new A.P.'s arisen from sub-sequences of A.P.'s. This indicates that the present instructional programme could not improve students' performance in these aspects. However, such weaknesses might be related more to the general training in solving mathematical problems than to specific knowledge of the topic.

Another interesting observation is that two students attempted to apply proportion to terms in an A.P. in Problem 4 (see Table 12). That is to say, they tried to write down something like $\frac{T_m}{T_n} = \frac{m}{n}$ when they could not find any way out otherwise. This idea of ratio was never observed among the Form 5 students in the pilot test. Nevertheless, these two students did not apply the idea of proportion in other problems, say Problem 5. This may be explained by the difference between the two problems. The information in Problem 4 could fit into a proportion more easily than those of Problem 5. It may be speculated that students in the lower grades were more likely to assume that unfamiliar quantities of similar nature would bear proportion to each other because they did not know much about other mathematical relationships at this stage. Alternatively, these two students might have actually associated terms in arithmetic progressions with proportion in their conceptual understanding. This point has to be left for further exploration.

Analysis of Problem-Solving Protocols in Task-based Interviews

Protocols from task-based interviews provided the raw data for inferring knowledge structures and studying problem-solving behaviours of the interviewed subjects. Before presenting the results obtained, the procedures taken in this analysis are described first to illustrate the extent to which the results were grounded on the primary data.

Six students were selected from each of the control and the experimental groups on the basis of their mathematics achievement test and mathematical reasoning test scores. Three students had high scores in both tests and another three students had moderately low scores in these tests. (Students of very low scores might not provide much data in thinking-aloud sessions as the pilot interviews had shown.) Their problem-solving test scores corresponded to the good and the poor performance divisions respectively. However, it should be noted that problem-solving test scores for the low performance division in the experimental group were generally higher than their counterparts in the control group. This has already been demonstrated in the statistical analyses in previous sections. (Again, in subsequent discussion, the control and the experimental groups are referred to as group A and group B respectively.)

Protocols of these subjects were analysed to provide information on how the basic concepts were conceived, how problems were represented, in what order the relevant variables were thought of and how the solution process proceeded. Some verbal answers directly revealed subjects' conceptual knowledge. For most of the items, tree diagrams were drawn to represent how a problem was solved by the subject. This would reveal the use of procedures and formulas. The problem-solving process could also be classified accordingly. Answers to retrospective questions helped to infer processes which were not verbalized, though not with complete certainty. Other relevant clues to the analyses also included underlined or circled words in the questions, key numbers marked on the script, lines and dots drawn, etc. These were all taken into consideration in making inference. Results obtained from one problem could be further evidenced by results in other problems. Hypotheses made in analysing one item might be later rejected by results in another items.

Based on information gathered from all the items attempted, the knowledge structure of the subject would be inferred. Besides, the problem-solving behaviours of the subject was closely observed.

An example of protocol analysis. To illustrate the procedures, the analysis of a sample protocol together with the subject's written solution will be described. For convenience of presentation and reference, protocols were typed in numbered lines. Dots were added to indicate short periods of silence. Other comments would be inserted by square brackets to make it more intelligible. Notice that the verbalizations made by subjects were originally in Cantonese usually with English mathematical terms and were therefore transcribed in Chinese. But, for the sake of presenting results here, extracts of protocols were translated into English.

Consider the sample protocol of subject A36 in Figure 5. (The code number A36 indicates that the subject belonged to Group A, the control group.) Problem 15 depicted involves the summation formula as an equation which is a key step to solution. It is observed that the presentation of the first term, the last term and the sum triggered off the formula with n as an unknown (line 2). The subject, pausing for a moment upon reading the problem statement, wrote down the equation readily. Later, n was related to the number of terms which entered into the second procedure to find d (line 7). However, the subject paused and remained silent when n was solved (lines 5-6). To clarify the silent steps, restropective questions were asked by the interviewer (I) and the subject's (S) answers were as follows:

I: What prompted you to write down the first equation here ?
S: I don't know ... I just think of this formula which connects all these numbers.
I: Why did you pause when you got n = 14 ?
S: I am thinking of how to go on.
I: You don't know the use of this value ?
S: Not before I am aware that it represents the number of terms.
I: So, you don't have a clear plan at start.
S: No.

Item 15 in Task-based Interviews

An A.P. 42, ... , 133 has a sum of 1225,

i.e.

$$42 + \dots + 133 = 1225.$$

Find the common difference of this A.P.

Protocol of Subject A36 (High Ability Group)

1. [Read the question.] Sum is 1225....
2. 1225 is equal to n over 2 times 42 plus 133.
3. [Write down the equation.] 1225 is n over 2 times .. 175.
4. n is equal to ... [an expression by transposing terms].
5. [Use the calculator.] n is 14.
6. ... n is 14. That is, 133 is ...
7. the 14th term. 14 terms ... 13 ... 13 differences.
8. So, the common difference is 133 minus 42
9. divided by 13.
10. [Use the calculator.] d is 7.

Written Solution

$1225 = \frac{n}{2}(42 + 133)$	(line 2)
$1225 = \frac{n}{2}(175)$	(line 3)
$n = 2 \times \frac{1225}{175}$	(line 4)
$n = 14$	(line 5)
$\frac{133 - 42}{13} = 7$	(lines 8-9)
$d = 7$	(line 10)

Tree Diagram

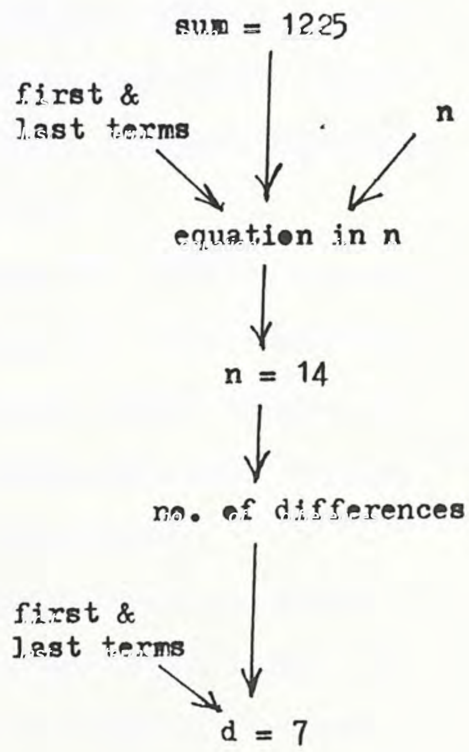


Figure 5. Analysis of a sample protocol and written solution collected in a task-based interview

Based on the thinking-aloud protocol, the written solution and answers to the retrospective questions, the following results and inference were obtained:

1. The summation formula was used correctly.
2. This formula could be triggered off by some of the variables involved. The word 'sum' might be essential in this step.
3. The number of terms and the number of common differences in between were related correctly.
4. The common difference could be calculated readily without using the general term formula.
5. The problem representation was presumably in the usual algebraic notations and the variables involved were solved as an algebraic problem. When more was known about the problem (when n was solved), the situation was perceived to be more concrete so that the solution could proceed through a concrete representation of an A.P.
6. The meaning of the algebraic symbols might not be consciously aware at every stage of the solution process. That is, the procedures used might be executed automatically until interpretation became necessary for relating to the situation.
7. This solution process was classified as forward searching because the subject worked from the givens but without a clear plan in mind before some values were interpreted. (No backward reasoning was observed however.)

Referring to the standard knowledge structure in Figure 3, we could conclude that some specific knowledge items were observed. For example, this included the first summation formula which could be used as an equation and the relationship between the number of terms and the number of common differences in between. The association between these knowledge items were inferred from the way the subject related these items in different tasks. For example, the summation formula was observed to be triggered by the key word 'sum'. In constructing the knowledge structure for subject A36, it was also noticed in other items that he seldom used the summation formula shown here, and the alternative formula was preferred. With regard to the subject's problem-solving performance, less search was involved in simpler problems and the problem representation was such that the meanings of algebraic symbols were quite clear from the start.

Sample protocols for the whole sessions of two subjects, one from each of the high and the low ability divisions respectively, are included in Appendix F. As a matter of fact, it was noted that Form 3 students in general found more difficulty in verbalization than Form 5 students interviewed in the pilot study. They were less capable in describing verbally what they were thinking of or working at. They were less aware of the reasons why some procedures were being executed or how they came to solve problems in some specific ways. This might be because most of them were not yet conscious in choosing methods to solve problems at this stage of learning. The verbalization was particularly poor with some subjects in the low ability division. Researchers have to consider these when planning future studies with younger subjects.

In the next few sections, results from the protocol analyses will be presented and discussed. Nevertheless, owing to the small number of subjects chosen, the comparison made upon the inferred knowledge structure and problem-solving performance may not be generalized in every aspect to the whole sample. In fact, qualitative differences in problem-solving strategies and knowledge structures certainly existed among any two subjects. The following descriptive analyses should be taken to indicate only some trends and prominent features.

Inferred Knowledge Structures of Interviewed Subjects

The knowledge structures for twelve subjects were constructed by analysing their protocols in task-based interviews. The basic knowledge items mentioned in the standard knowledge structure were assessed. Table 13 summarizes the main features observed in their inferred knowledge structures. Relationships between these knowledge items, if observed, were primarily the same as those depicted in the standard knowledge structure (see Figure 3). Of course, when some items were missing from the structure, relationships would be modified.

Conceptual knowledge. The basic terminologies were properly used and understood by almost all subjects. Only, two of them had shown uncertainty in determining differences with some special cases of sequences. All

Table 13

A Summary of the Features Observed in the Knowledge Structures of Interviewed Subjects

Items in knowledge structure	Good Performance						Poor Performance					
	A08	A33	A36	B16	B19	B27	A12	A26	A38	B02	B06	B17
Conceptual knowledge												
Definition of A.P.	*	*	*	*	*	*		*	*	*	(*)	*
Common difference	*	*	*	*	*	*	(*)	*	*	*	*	*
Distinction between k and T_k	*	*	*	*	*	*	*	*	*	*	*	*
Recognising sum of terms	*	*	*	*	*	*	*	*	*	*	*	*
Procedural knowledge												
$T_n = a + (n-1)d$ (α)	*	*	*	*	*	*	(*)	(*)		*	*	*
$T_m - T_n = (m-n)d$ (β)	*		*	*	*	*	(*)		*	*	*	*
$S_n = \frac{n}{2}(T_1+T_n)$ (γ)	*	*	*	*	*	*				(*)		
$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ (δ)	*	*	*	(*)	*	*	(*)			*	*	*
Use (γ) whenever T_n is given	*	*		*								
Use concrete representation for A.P. in formulating procedures	*		*	*		*			*		*	
Use a and d as two essential characteristics of any A.P.	*	*	*	*	*	*	(*)			*	*	
Use (α) as an equation	*	*	*	*	*	*	*	*	*	*	*	*
Use (β) as an equation	*		*	*	*	*					*	
Use (γ) as an equation	*	*	*	*	*	*						
Use (δ) as an equation			*	*		*						

Note. '*' indicates correct understanding or correct formula. (*) indicates uncertainty in understanding or wrong formula.

subjects could mention the pattern of constant difference for an A.P. But later items revealed that the meaning of "difference" varied (Items 2 and 3 in Appendix E). Apart from the correct idea of formal subtraction, some included the condition of all terms being of the same sign. Two subjects considered 5, 55, 555, 5555 as an A.P. and gave the "common difference" as 5. This indicates that a conceptual error might be very subtle. In manipulating <u>given</u> arithmetic progressions, the error might not surface. It also shows that some students at this stage might not be able to work strictly on a formal level. Peculiar examples could cause instability in their conception.

All subjects interviewed could distinguish clearly between the order of a term and the term itself. This would presumably be confused when one of them was an unknown. For instance, in Problem 2 of the written problem-solving test, a few subjects made this mistakes (see Table 12). Of course, this confusion would cause errors in further calculations. It was also observed that all of them could discriminate terms and the sum of terms in the given problem situations.

Since the relationships involved in this topic were quite simple and straightforward without much conceptual ambiguity, the subjects could relate and state the variables in the proper ways in interpreting the tasks or answering questions. In fact, it is improbable that the subjects could build up other associations after the instruction because this topic was quite detached from other areas of mathematics apart from simple algebra and arithmetic. Minor variation was found for the increasing/decreasing property of A.P.'s. Most subjects consciously related it to the sign of the common difference as revealed by their protocols. But two subjects considered it solely as a condition for checking A.P.'s. This was not wrong, but the consequence might be that for A.P.'s given, this property would not be utilized in relevant ways.

Procedural knowledge. The use of formulas observed in the protocols followed similar trends as found in the analysis of the written test (compare Table 11). The general term formula was known correctly to 3/4 of the interviewed subjects. The summation formulas were again more difficult. These formulas could be used by about half of them as equations. The concrete representation was used by about half of the subjects in formulating the problems. This was evidenced by the way they wrote the sequences or they counted the intervals between terms. About 3/4 of them recognized that in solving problems with A.P.'s, the first term and the common difference were two essential characteristics which, if known, could be very useful.

In working with formulas, about half of the students could only use them to compute the unknown on the left side but could not apply them to write equations (see Table 13). This may be interpreted as either poor general algebraic knowledge or unfamiliarity with the topic. The latter possibility might cause hindrance to students in handling the variables freely as they could normally do in other familiar area.

As far as these twelve subjects were concerned, wrong formulas did not occur too often. But as for the two summation formulas, many subjects could only use one of them and forgot the other entirely. This point has been noticed in analysing the written test. The conditions of application of the two formulas slightly differed. It had been brought out in the instruction explicitly. Students might not have recognized this point and simply choose to memorize only the more general one. Alternatively, it may be speculated that students might believe that it was sufficient for them to get one way to work out something.

Nevertheless, among all interviewees, the item on the property of summation was not observed to relate to the summation formulas. This was probably because after using this property in proving the summation formulas in the instruction, it was not mentioned in any other applied context. But, as students usually ignored the details of proof, they did not notice the significance of the property in understanding the formulas. Alternatively, the algebraic step that connected the property to the formulas might not be understood properly. Perhaps, if this property is to be acquired in future instruction, it should be applied in some practical cases other than the proof. The link between the property and the formulas should be given more emphasis too.

<u>Two sample inferred knowledge structures</u>. To give a lucid picture of the knowledge structures obtained,

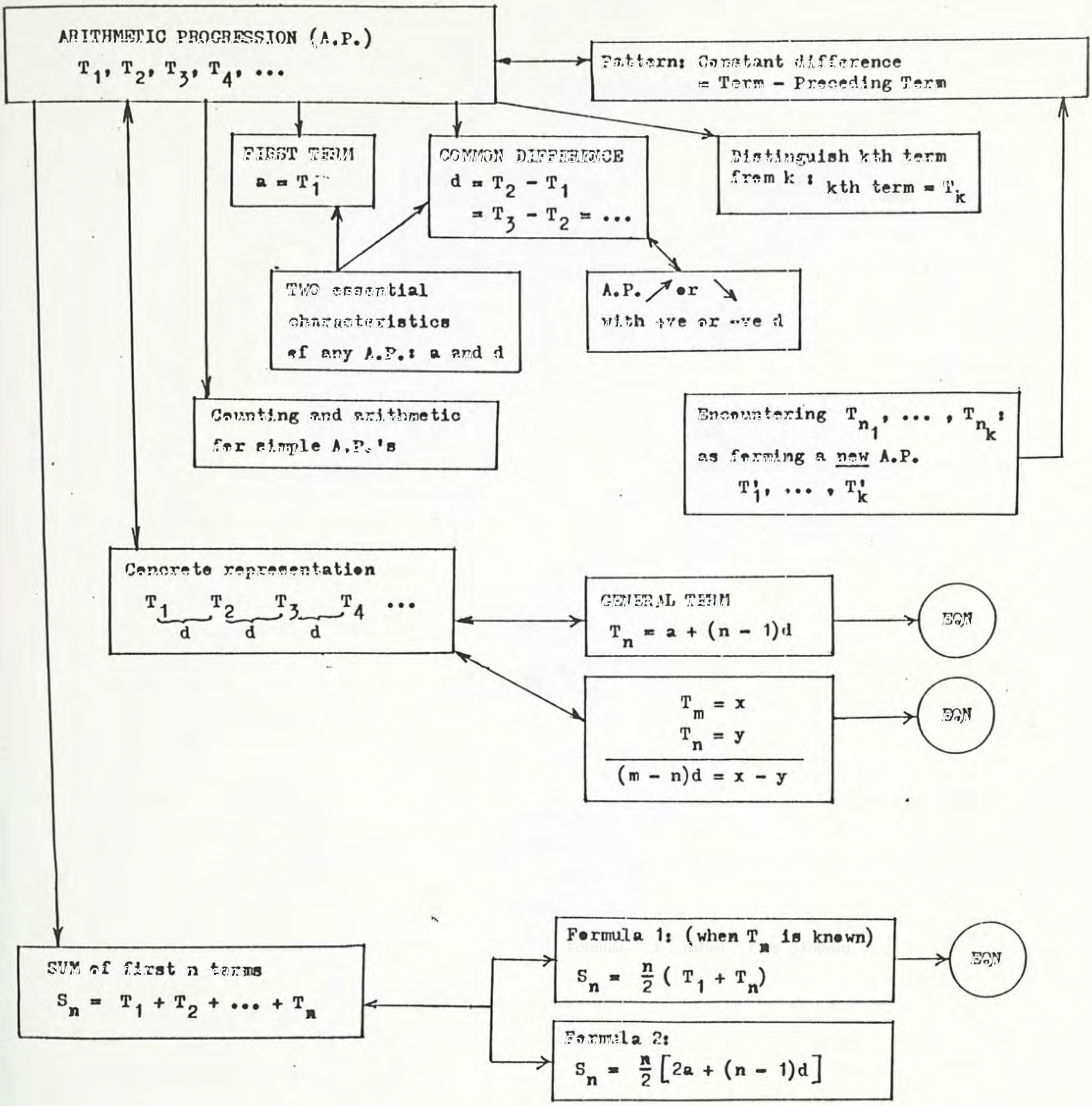
Figures 6 and 7 present the inferred knowledge structures of two subjects as revealed by their protocols and written solutions. These may be compared with the standard knowledge structure in Figure 3. Subject A08 belonged to the high general ability group and performed very well in the problem-solving test. Subject A12 belonged to the low general ability group and got low scores in the problem-solving test. (Protocols of these two subjects are included in Appendix F.)

In Figure 6, it can be clearly observed that subject A08 possessed a knowledge structure accurate in conceptual understanding, containing standard formulas which could also be executed to form equations and resembling closely to the standard knowledge structure. In fact, subject A08 developed her own way of dealing with the problem given T_m and T_n, a written form slightly different from that mentioned in the instruction and perhaps more susceptible to memory (compare Figure 3). She employed this procedure in solving many problems. The concrete representation for an A.P. was confirmed by the following protocol in handling Item 7:

<u>Item 7</u> If 27, x, y, z, 39 is an A.P., find x, y and z.

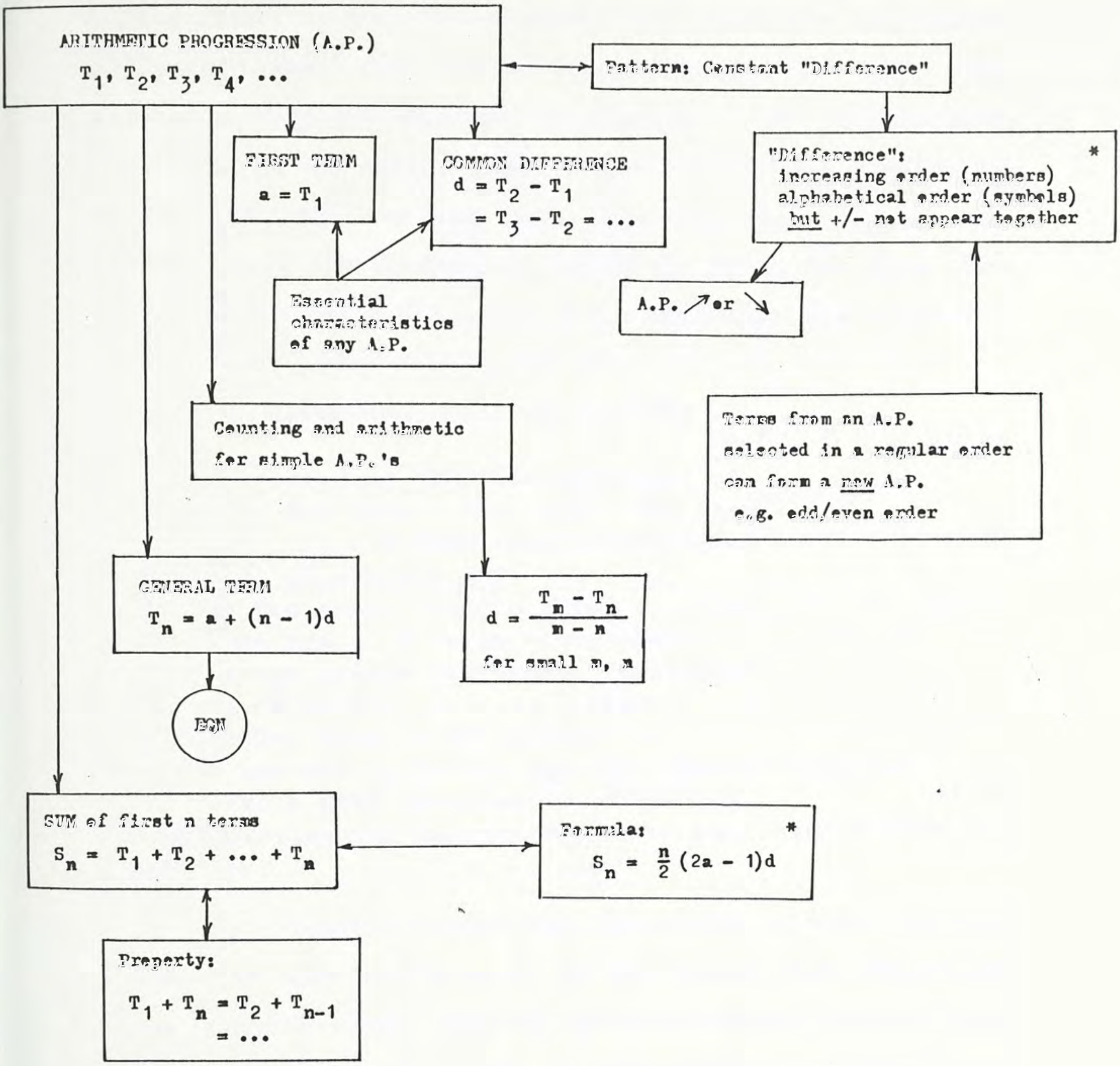
<u>Protocol of subject A08</u>

1. [Read the statement.] in A.P...
2. [Point to the space between the numbers and
3. unknowns and count.] one, two, .., four
4. 4d is 39 minus 27. [Write the equation.]
5. d is 3.
6. So x is 27 + 3 = 30.
7. y is 30 + 3 = 33.
8. z is 33 + 3 = 36.



EQN : The formula can be applied to set up equations.

Figure 6. Inferred knowledge structure of subject A08 (high ability group).



Note. Items with asterisks are wrong ideas or wrong formulas.
EQN means that the formula can be applied to write equations.

Figure 7. Inferred knowledge structure of subject A12 (low ability group).

The sequence shown was perceived by the subject in the form T--T--T--T--T where '--' stood for one common difference when she counted them (lines 2-3). When sub-sequences were encountered in Items 8 or 18, the subject was observed to relate readily the new common difference to the original one and calculate the number of terms immediately. For instance, in Item 8(d), we have the following protocol and lines 2-4 illustrate the immediate procedure of processing a new A.P.:

<u>Item 8</u> Given an A.P. 3, 9, 15, 21, ...

....

(d) Find the following sum:

3rd term + 6th term + 9th term + ..

... + 15th term + 18th term

<u>Protocol of subject A08</u>

1. The 3rd term, the 6th term, ..., yes, an A.P. again
2. There are .. 6 terms altogether.
3. The new common difference is greater...
4. should be 3d, ... i.e. 3 times 6 , 18,..
5. The 3rd term is 15 [given]
6. So the sum is [write down the second summation
7. formula that involves a, d and n]

Her calculation was correct and this part was omitted for simplicity.

Figure 7 shows the knowledge structure of a subject of poor performance in the topic. Some knowledge items contain wrong idea or wrong formulas. Subject A12 attempted up to 17 items in the interview (20 items available). It was observed that she held more restrictions to test A.P.'s. In fact, in distinguishing A.P.'s, she rejected the sequence -3, 1, 5, 9, 13 (Item 2(c)) to be an A.P. by the reason that there were both positive and negative numbers. For sub-sequences, she certainly did not understand the underlying conditions. In

Item 8(d), when she was asked why she said the given sequence was an A.P. (this same item has just been mentioned for subject A08), she gave the following answer:

S: These numbers [refer to the suffixes] are multiples of 3.
I: Yes, these are multiples of 3. But that means ...
S: These show a regular pattern.
I: What pattern?
S: ... even or odd numbers or multiples
I: How about T_3, T_8, T_{13}, T_{18} [writing down] ? Is it an A.P.?
S: .... of course not.

Subject A12 used simple arithmetic and listed the sequence exhaustively in many cases. Only the general term formula was known to her. When manipulating terms, the relationship between T_m , T_n and d was used only for terms with values given and small m and n's. For summation of A.P.'s, it is interesting that she could start by adding the first and the last terms, the second and the second last terms, but could not go on if there were too many terms to be added. So she knew this property of the summation of A.P.'s. But she could not utilize this property effectively to relate to the summation formulas, indicating that she was weak in general algebra. For sums of many terms, a wrong formula was employed instead (see Figure 7), and gave results which seemed absurd to her too. For instance, she did expect sums to be integers in most cases but the wrong formula did not do so in some cases (when the numerator was an odd number).

When the knowledge structures of subjects A08 and A12 just described are compared, differences can be observed in the following aspects:

1. the accuracy in understanding basic terms,

2. known properties related to these items,

3. the availability of procedures and formulas,

4. relationships between items,

5. the closeness to the standard knowledge structure.

Briefly to say, subject A08 possessed a knowledge structure that was more close to the standard knowledge structure (cf. Figures 3, 6 and 7). Basic notions were correctly understood. More procedures and properties were known and more proper relationships were built up between these items.

Differences in the inferred knowledge structures between students of good and poor performance. When subjects were divided into two groups in terms of their performance in the problem-solving test, the differences in their knowledge structures could be compared as shown in Table 13. Differences were found to be significant in terms of the accuracy of conceptual understanding and the knowledge of procedural rules.

Subjects in the poor performance group tended to include other surface features of the sequences in checking for A.P.'s. The concept of "difference" might be interpreted in ways other than formal subtraction. In terms of the knowledge of formulas, subjects in the poor performance group were only moderately competent with manipulating terms and were very weak in working with summation. Most subjects in the good performance group could use the formulas in setting up equations. But, this was only the case with the general term formula in the poor performance group. Furthermore, only a few subjects in the good performance group showed the knowledge of the conditions for choosing between the two summation formulas, but none in the poor performance group. As noted previously, in both good and poor performance groups, the item on the property of summation was not observed to relate to the summation formulas.

While Table 13 cannot show the relationships between the knowledge items, the two sample knowledge structures discussed in the last section in fact demonstrated the typical differences found in knowledge structures between the good and the poor performance groups. Referring to Figures 6 and 7 as examples, we can see that subjects of the good performance group possessed knowledge structures with more relationships built up between knowledge items and the overall structures were more close to the standard structure so that the organization of concepts and formulas was more coherent. By contrast, the knowledge structures of subjects in the poor performance group were less similar to the standard structure and less coherent. For instance, in the knowledge structure of subject A12 (Figure 7), items such as the one on the concept "difference" and the one on $(T_m - T_n)$ were related to other items with improper understanding of procedures or concepts.

These differences were as expected and were in close agreement with the results from other expert-novice studies. To account for these differences, it is simply to note that on the one hand, the knowledge structure was inferred from the problem-solving performance. On the other hand, this hypothesized construct directly influenced problem-solving. A problem encountered would be represented in an appropriate form by retrieving relevant concepts and rules. The readiness and effectiveness in solving problems among students in the good performance group were attributed to the correct conceptual understanding and the rich relationships existed in their knowledge structures. It was also noticed that the more relationships found in the knowledge structures, the less possible that knowledge items held would be inconsistent with each other.

<u>Differences in the inferred knowledge structures between students of the control group and the experimental group</u>. When the twelve subjects were divided into two groups in terms of the instructional method, i.e. group A and group B, the variation in knowledge structures within either group and between the groups could also be seen from Table 13. That is, subjects of code numbers starting with A are compared with those starting with B.

As discussed in the last section, the good performers in both groups had similar coherent knowledge structures with rich networks of relationships. Although there were some differences such as the condition for applying the summation formula, these differences in minor details should not be magnified to an inappropriate extent when the small number of subjects involved was taken into account. Briefly to say, knowledge structures did not differ significantly for the good performers in the control and the experimental groups with regard to conceptual knowledge, formulas and procedures and relationships between these items.

On the other hand, when subjects in the poor performance range were compared, subjects in the experimental group (group B) showed a significantly higher familiarity with the formulas. They could more readily execute these formulas in direct computation. More subjects in group B grasped the first term and the common difference as two essential characteristics that were useful in approaching problems. Poor performers in the control group (group A) were quite weak in using the general term formula and almost unable to recall the summation formulas. However, relationships between conceptual knowledge items were more or less similar.

These findings supported the hypothesis that the use of structural diagrams had positive effects on the acquisition of knowledge structures. However, it cannot be claimed that knowledge structures in the experimental group were more close to the standard knowledge structure. Effects were only observed on the acquisition of the few formulas which were related to the basic notions of the topic. The relationships found in knowledge structures of the two groups were quite similar. Structural diagrams did not seem to improve the understanding which was an important element in constructing relationships between different items of knowledge. This was evidenced by the

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absence of using the concrete representation of an A.P. in solving problems for half of the interviewed subjects in both groups altogether (see Table 13). This representation had been deliberately introduced in the instruction.

Nevertheless, it was observed that subjects in the low general ability division of the experimental group were gaining more from the instruction compared with those in the high ability division of the same group. They were more familiar with the standard formulas compared with subjects in the low ability division of the control group. So, a weak interaction effect of the instructional method and the ability level on the acqusition of knowledge structure was actually detected.

In retrospect, although structural diagrams could illustrate the structure of problems or display the relationships between different knowledge units, it might not be effective enough to foster understanding among less able students who were presumably not capable of perceiving structures even when concretely represented. This may explain why the effects observed above were so limited.

From the two comparisons made, we can conclude that subjects of the good performance group possessed knowledge structures significantly different from those of the poor performance group in terms of conceptual understanding and knowledge of procedural rules. Their knowledge structures contained more relationships between items and resembled more closely to the standard knowledge structure. The use of structural diagrams had positive effects on the acquisition of knowledge structures, but the effects were only observed on the acquisition of the standard formulas and procedures. There was a weak interaction effect of the treatment and the mathematical ability on the acquisition of knowledge structures, favouring the low ability division of the experimental group.

<u>Problem-Solving Performance Observed in Task-based Interviews</u>

As discussed in a previous section, analyses of the thinking-aloud protocols together with the written solution yielded information which revealed the subject's problem representation and problem-solving strategies employed. Observations in these two aspects will be discussed first before making comparisons.

<u>Problem representation</u>. When a problem was encountered, the subject would produce his or her own way of looking at the task through reading the problem statements. This involved the understanding of the problem situation as perceived by the subject. In the thinking-aloud protocols, the problem representation could be inferred from observing the meaning attached to the given words or information by the subject and the way the problem was related to the usual terminologies of the topic. The problem representation played a key role in the solution process because the subject would choose

strategies according to how the task was interpreted.

Before the classification of problem representations are discussed, the types of problems used should be noted first. Items used in the interviews were mostly problems set in an algebraic form (see Appendix E). Only a few of them were applied problems. With problems of the algebraic type, the terminologies of the topic were used explicitly in the problem statements. With the applied problems --- only 4 such items in the batch (Items 14, 17, 19 and 20), terminologies of arithmetic progressions were not used except in one item (Item 19). The words used in the problem statements would influence the problem representation employed by the subjects.

Primarily, three different types of problem representations were identified from the protocols. They were named as follows: elaborated algebraic representation, formal algebraic representation and arithmetic representation. This classification was based on the empirical data obtained and would be restricted for problems used in the task-based interviews only. For problems in other area of mathematics, this might have to be expanded.

An arithmetic representation refers to an understanding of the problem components as numbers with some kind of numerical relationships. A formal algebraic representation involves the usual algebraic notations as used in the instruction to represent the elements in the problem. But, the understanding of these elements remains abstract and relationships perceived by the subject are restricted to those given by standard formulas. An elaborated algebraic representation, apart from using the usual algebraic notations, contains further elements corresponding to more understanding pertinent to the problem situation, or relationships tied to the problem at hand specifically.

An arithmetic representation can be illustrated by the protocol for Item 12 shown in Figure 8. Subject A38 talked about the given A.P. as simply a list of numbers with the pattern of constant difference (lines 5-7 and 10). All the problem-solving attempts worked through the given numbers or other numbers generated by subtraction or addition (lines 7 and 11). Although "A.P." and "common difference" were mentioned in the protocol (lines 2,3,5 and 6), the subject did not relate these further to the formula he knew. He did not even mentioned that n was the unknown. He could not perceive n because it was not concretely hinted in the problem. He could only see a sequence of numbers. From this protocol, an arithmetic representation was inferred. Note that even with an arithmetic representation, Item 12 could still be solved if the numbers were actually being listed. However, we can say that this representation was not effective enough to solve many other problems.

The formal algebraic representation was used most often because it only involved the introduction of formal algebraic notations. This representation did help subjects to relate the givens and unknowns to the standard

Item 12 in Task-based Interviews

Find the sum of the following A.P.:

8 + 15 + 22 + ... + 78.

Protocol of Subject A38

1. [Read the question.] Find the sum.
2. A.P. ... How much is the sum?
3. To find the A.P. first.. Yes..
4. Find the terms? No. Find the sum.
5. This is an A.P... 8, 15, ...
6. 7, yes 7 is the common difference
7. 78 minus 7 is 71 .so. 71, 78 .....
8. But how many terms are missing? ...
9. Many. Have to list out to add.
10. 1st, 2nd, 3rd terms we have already..
11. Next is 29 [start using calculator]..
12. But can we reach 78? Clumsy way.
13. [Give up the listing.] Any other way?
14. .... [Other unsuccessful trials using
15. the calculator to multiply or divide.]

Figure 8. Protocol of subject A38 illustrating an arithmetic representation of Item 12.

procedures and formulas so that relevant strategies might be retrieved. Figure 9 presents a protocol for Item 7 which was classified as a formal algebraic representation. According to the way subject B16 wrote down the equation

$$39 = 27 + (5 - 1)d \,, \qquad \text{(lines 2-3)}$$

it is reasonable to infer that a formal correspondence between the problem elements and the algebraic formulation was set up by the subject as follows:

$27 \longleftrightarrow a$

$5 \longleftrightarrow n$

$39 \longleftrightarrow T_n$

$d \longleftrightarrow d$ (unknown in this problem)

so that the equation could be written readily with the formula $T_n = a + (n - 1)d$ in mind (lines 2-3).

<u>Item 7 in Task-based Interviews</u>

If 27, x, y, z, 39 is an A.P., find x, y and z.

<u>Protocol of Subject B16</u>

1. [Read the question.]
2. ... 39 is the 5th term.
3. which is 27 + (5 - 1)d. [Write an equation.]
4. 4d = 39 - 27 [Solve the equation on paper.]
5. .. 12 ... d is 3.
6. So x = 27 + 3 = 30 [Write this addition.]
7. y = 30 + 3 = 33 [Write this.]
8. z = 33 + 3 = 36 [Write this.]
9. O.K.

<u>Figure 9</u>. Protocol of subject B16 illustrating a formal algebraic representation of Item 7.

This example in Figure 9 was a correct representation of the problem which led to the final solution. However, this might not be always the case. Students were also observed to use the formal algebraic representation without clearly understanding the problem. Figure 10 illustrates such an example. Subject B06 did notice that numbers not divisible by 3 were to be added (lines 2, 4 and 8), but when he determined to use the summation formula, he gave up the attempt to investigate

Item 14 in Task-based Interviews

Find the sum of all the numbers from
1 to 60 which are NOT divisible by 3.

Protocol of Subject B06

1. [Read the question.]
2. Sum of all numbers .. NOT divisible by 3.
3. All numbers between 1 and 60 .. Sum.
4. ...... NOT divisible by 3 .... Total..
5. ..[n times] 2a plus n minus 1 times d over 2
6. Ay. What is n? 60... But d should be ..
7. Now, I must find the 1st term first...
8. .... numbers not divisible by 3...
9. Yes, the first one is 1. Let's try.
10. 60 numbers .. 2 times 1 ..plus 60
11. minus 1 times d divided by 2 [Write this.]
12. to find the sum ....
13. But d is still unknown.
14. Is it 3? .. [Substitute and compute.]
15. The answer is 5370.

Figure 10. Protocol of subject B06 illustrating an incorrect formal algebraic representation of Item 14.

these numbers and proceeded to relate the algebraic formula to the numbers associated with the problem (lines 9-15). Although each of the algebraic symbols eventually received an input (lines 6, 9 and 14), the answer obtained was in fact meaningless (line 15). In spite of the incorrect correspondence set up between the problem elements and the algebraic formulation, it was classified as a formal algebraic representation.

An elaborated algebraic representation was inferred when the subject's protocol contained clear statements that the specific problem situation was at the centre of consideration apart from the algebraic formulation employed in expressing the problem elements. Figure 11 presents a sample protocol for Item 10 which was classified as an elaborated algebraic representation. It is observed that subject A08 worked with the usual algebraic formulation (note a and d in lines 3, 4 and 7), but with deep understanding of the elements involved. She found the common difference readily by mental calculation (line 3). Then she proceeded to <u>transform</u> the given A.P. to another A.P. with new indices (lines 6-7):

T_6 <---> T_1'

T_{27} <---> T_{21}'   [should be T_{21}']

Although strictly speaking, she had written the notation and one index wrongly (using the same T and 22 instead of 21), her good understanding could not be denied. She worked very flexibly with the information given throughout and was not tied to the formal algebraic expressions. This contrasted with the formal algebraic representation. For

Item 10 in Task-based Interviews

In an A.P., the 6th term is 20,
and the 11th term is 50.
Find the 27th term.

Protocol of Subject A08

1. [Read the question.] O.K.
2. The 6th term is 20, the 11th is 50.
3. Then 5d is 30 [by mental computation 50 - 20].
4. .. d is 6. [Write down d = 6.]
5. That is, 27 minus 6 is 21.
6. [Write down T_{21}.]
7. equals a .. a is 20 [This is T_6.] ..
8. plus 20 [this is (21 - 1)] times 6 ..
9. which is 120 plus 20 is 140.
10. [The following expression is written
    $T_{21} = 20 + 20 \times 6 = 120 + 20 = 140$.]
11. So, T_{27} is 140.

Note that the answer obtained was wrong. Subject A08 had made a slip when she wrote T_{21} (line 6). It should be T_{22}, in the new A.P. perceived. The following questions and answers after the subject finished should also be considered.

I: Why did you say a is 20?
S: This [pointing to T_6] can be treated as the first term for calculation.
I: Then....
S: The 27th term to be found would be the 21st term in this new A.P.
I: I see. So you write T_{21} here?
S: Yes.

Figure 11. Protocol of subject A08 illustrating an elaborated algebraic representation of Item 10.

Item 10 in Task-based Interviews

In an A.P., the 6th term is 20,
and the 11th term is 50.
Find the 27th term.

Protocol of Subject B17

1. [Read the question.]
2. The 6th term is 20.
3. That is, a plus 5d is 20.
4. [Write an equation a + 5d = 20.]
5. Ah. a plus 10d is 50.
6. [Write an equation a + 10d = 50.]
7. .. So. solve it.
8. 10 minus 5 is 5. 5d is 30 [by subtraction]
9. [Write the equation 5d = 30 and]
10. d is 6. [Write the solution.]
11. Substituting this, a is 20 minus 30.
12. [Write a = 20 - 30 = -10.]
13. So a is -10.
14. ... To find T_{27}....
15. [Continue.] T_{27} is a [writing -10]
16. plus 26 times d [writing 26 x 6]
17. [Use calculator.] is 146. The answer.

Note that the answer obtained was correct. Subject B17 followed closely the algebraic formulation throughout and the information given in the problem was not attended to again once the correspondence between the problem elements and the algebraic notations had been established.

Figure 12. Protocol of subject B17 illustrating a formal algebraic representation of Item 10.

instance, Figure 12 presents the protocol of another subject for the same item which was only classified as a formal algebraic representation. The figure should be self-explanatory.

Table 14 tabulates the results of classifying the problem representations employed by the subjects in tackling the items in the task-based interviews. It should be noticed that the particular <u>type</u> of problem representations used did not guarantee success or failure in solving the problem. The correctness of the solution can be found in Table 15 (to be discussed later). This has been emphasized in the previous examples cited. The crucial point is that the type of representations employed reflected the understanding of the problems in relation to the knowledge possessed in this topic. Of course, the arithmetic representation was very limited in scope if applied to problem-solving. A formal algebraic representation was usually sufficient to facilitate solution provided that it was correct and the subject possessed the knowledge of the relevant procedure. An elaborated algebraic representation would be associated with elegant solution methods that reflected a good understanding of the specific area being tested.

As Table 14 shows, the formal algebraic representation was employed most often, over 2/3 of the problems attempted. The arithmetic representation was mainly found among subjects of the poor performance group. It was also employed by others in novel problems. The elaborated algebraic representation was mainly used by

Table 14

Problem Representations Employed by Subjects in the Task-based Interviews

Items in task-based interviews	High Ability Good Performance						Low Ability Poor Performance					
	A08	A33	A36	B16	B19	B27	A12	A26	A38	B02	B06	B17
Simple problems												
5.(a)	B	B	A	B	B	B	A	A	A	B	A	B
(b)	B	B	A	B	B	A	C	A	A	A	B	A
(c)	B	B	B	B	B	B	-	B	B	B	B	B
(d)	B	B	B	B	B	B	-	B	-	B	B	B
6.	B	B	B	B	B	B	C	C	C	B	B	B
7.	A	B	A	B	A	B	A	C	C	B	B	B
8.(a)	B	B	B	B	B	B	-	-	B	B	B	B
(b)	B	B	B	B	B	B	C	C	C	B	B	B
(c)	B	B	B	B	B	B	-	C	C	B	B	B
(d)	A	A	C	A	A	A	-	C	C	B	C	B
9.	A	B	A	A	A	B	C	B	C	A	B	B
10.	A	B	A	A	A	B	B	B	C	B	B	B
Harder problems												
11.(a)	A	A	A	A	A	A	-	C	C	B	B	B
(b)	A	A	A	B	A	B	-	B	-	B	-	B
12.	B	B	B	B	B	B	C	C	C	B	B	B
13.	A	A	A	A	B	C	C	C	C	B	C	B
14.	A	B	C	A	A	C	C	C	C	C	B	-
15.	B	B	B	B	B	B	-	-	C	B	B	B
16.	B	B	B	B	B	-	-	-	-	B	B	B
17.(a)	A	A	B	A	A	A	C	C	C	B	B	B
(b)	A	A	A	A	A	A	C	C	C	B	B	B
18.(a)	B	B	A	A	B	A	-	-	-	-	B	B
(b)	A	B	A	A	B	A	-	-	-	-	B	B
19.	B	A	A	A	B	A	-	C	-	-	B	B
20.(a)	B	A	C	B	B	-	-	-	-	-	-	-
(b)	-	B	C	-	-	-	-	-	-	-	-	-

A: Elaborated algebraic representation

B: Formal algebraic representation

C: Arithmetic representation

subjects of the good performance group. Of course, in simple problems, subjects of the poor performance group might use it too.

From this table, it can be seen that each individual subject had a tendency to use certain types. For example, subject B17 always used the formal algebraic representation. Subject A38 used the arithmetic representation nearly 4/5 of the time. On the other hand, some problems were more susceptible to a particular type of representation. For instance, Item 16 was tackled by all subjects using a formal algebraic representation. This observation may be attributed to the particular context of the problem such as the elements involved or the difficulty level. To conclude, we can say that this table demonstrated that the problem representation was a function of the task as well as the subject.

One point should be noted before leaving the question of classifying problem representations. As discussed in the literature review, the problem representation may change during the problem-solving process according to the new understanding or information attended by the subject. In the present protocol analyses, it was noted that the details within the problem representation might change when the subject noticed some new relationships not perceived before. But, the type of representation was not affected. In some cases, the elaborated algebraic representation might be said to be gradually built up. However, since the items used were short problems, stages could not be distinguished and instead, the overall representation would be inferred from the data. Another possible reason for this lack of change in the representation types observed is that subjects from Form 3 classes were not competent enough in their metacognitive skills to control their attention during the problem-solving process so that they could only keep to one interpretation of the problem situation throughout except in minor details.

While the foregoing classification could be inferred with high certainty, it was relatively difficult to answer how the problem representation came about. The understanding process could only be inferred and was usually not verbalized by the subjects. For example, there was most often a period of silence after a problem was read. This period of silence was the crucial period when a representation of the problem was built up for such short problems used. Retrospective questions could not help much because rationalization after solution was highly probable.

As we have just concluded, the problem representation depended on both the task and the subject. The knowledge possessed by the subject certainly played an important role but how it effected a representation could hardly be overtly observed. The words or hints in the problem statements could be influential. The following general observation might support this hypothesis. It was observed that subjects could pick up some key words and interpret the givens and the goals without much difficulty for algebraic problems. They might mark these words and their values, if any, on reading the problem. These key words included "first term", "common difference", "kth term", and "sum". (The word "A.P." was not particularly noticed because it seemed to be assumed in all questions and was in fact redundant.) After setting the problem in the context of arithmetic progressions by employing the standard algebraic symbols, they would proceed to search for solution. This may provide a tentative explanation for the formal algebraic representation so often used with some problems (e.g. Items 5, 8, 10 and 16).

With those applied problems where terminologies of A.P. did not appear explicitly, the problem representation seemed to be constructed through a basic assumption probably held by all the subjects attended. That is, they took for granted that A.P.'s would always be involved in these problems. The general observation was that in such cases, students would extract sequences of numbers from the given information after comprehending the statements. They would then seek correspondence between the sequence of numbers and the parameters in the arithmetic progression formulation, using the usual notations. Sometimes, this process of generating a problem representation might be worked out by "brute force".

Thus, as discussed above, we can only provide these tentative explanations for the generation of problem representations. The present protocol analyses could not provide sufficient empirical data for inferring the exact process involved. This important process has to be addressed by future research.

Problem-solving strategies. Once an internal problem representation was constructed, the subject would proceed to solve the unknowns in some specific ways. Several types of problem-solving strategies were identified in the protocols. They were classified as follows: working forward, forward searching, backward reasoning, and trial-and-error.

The working forward strategy was usually found in familiar or typical problems, in which the subject started from the givens and attained the goal through procedures well-known to the subject (i.e. schema-driven problem-solving processes). For instance, the protocol shown in Figure 9 (refer to p.126) is an example of working forward strategy. The flow chart for the process is as follows:

$$\left.\begin{array}{l}\text{1st term= 27}\\ \text{5th term= 39}\end{array}\right\} \longrightarrow \text{Eqn. in } d \longrightarrow d \longrightarrow x \longrightarrow y \longrightarrow z$$

General term formula for T_5

The subject was sure that the equation would yield d which in turn would give the values of x, y and z directly. He proceeded to work in this direction without the necessity to reason (lines 2-3, p.126).

The forward searching strategy was also characterized by the subject working from the givens. But, in so doing, one generated some other variables and attempted to link these to the goal. The sample protocol and tree diagram in Figure 5 (p.104) clearly depict such a problem-solving process. After solving the problem, the subject was asked about the procedure he used. As his answer shows (refer to the retrospective dialogue on p.103), he did not have a clear plan to find the unknown until some newly computed values became useful.

Another example can be found in Figure 12 (p.130). When subject B17 had found the values of a and d (lines 10 and 13, p.130), she paused for a moment to consider the situation. Then she continued to compute the 27th term using these values (lines 14-17, p.130). When she had finished and was asked why she paused at that point, she expressed that she knew that with two terms given, setting two equations to solve a and d usually worked. But, she just did not know the way until she actually reached there with a and d computed.

Backward reasoning was a problem-solving strategy with reasoning backwards from the goal to see if any relation can link up the givens to the goal. This was only observed in five cases by four subjects in the present study (see Table 15 on p.141; subjects A08, A33, B02 and B17). Figure 13 shows one of these cases. Subject B17 began to reason from the values necessary for the summation formula if the sum was to be calculated (lines 2-4). Then she asked herself how to find n (line 6). These steps taken clearly demonstrated that she used backward reasoning in this case.

Item 12 in Task-based Interviews

Find the sum of the following A.P.:

8 + 15 + 22 + ... + 78.

Protocol of Subject B17

1. [Read the question.]
2. 8 plus 15 plus .....to sum..
3. I don't know the number of terms.
4. Ah. .. the others we know.
5. [Probably refer to a and d.]
6. How to find n? Yes. Simple.
7. 8 plus n minus 1 times ..[write 8 + (n - 1)]
8. 15 minus .. [Do mental calculation.
9. Must be 15 - 8.] 7
10. [Complete the expression 8 + (n - 1)7.]
11. equal to ... yes 78
12. .... [Solve the equation 8 + (n - 1)7 = 78.]
13. n is 11.
14. O.K. I can sum now.
15. S is 11 times bracket 2a [write 8 for a]
16. plus 10 times 7 .. over 2
17. [Write S = 11x[2x8 + 10x7]/2.]
18. [Use the calculator.] S is 473.

Remark. The reasoning process and the answer obtained were all correct.

Figure 13. Protocol of subject B17 illustrating the use of backward reasoning strategy in Item 12

The trial-and-error strategy might also work sometimes because the numbers involved and the answers to many test items were simple integers. In fact, some problems could be reduced to arithmetic problems with repeated addition or subtraction. Subjects could try out a reasonable guess to see if it worked. Figure 14 illustrates one such case. The protocol should be self-explanatory here.

<u>Item 16 in Task-based Interviews</u>

An A.P. has a common difference of 4.
Its 10th term is three times the 5th term.
Find the sum of the 11th term to the 16th term.

<u>Protocol of Subject B06</u>

1. [Read the question.] ....
2. .. An A.P. has common difference 4 ...
3. The 10th term is 3 times the 5th term..
4. 4 times 3 is 12.... What is the 10th term?
5. .... 10 minus 5 is 5... [Use the calculator]
6. 2.4 .... no way out...
7. ... 3 times the 5th term... that is
8. the 10th term ....equal to 5 .. no..
9. An A.P. has common difference 4 ...
10. that is, to find 5th term... that is
11. 10th term divided by 3 .... [pause]....
12. 5.. 5 minus 4 .. no.. the 5th term minus
13. the 1st term times 4 .... no...
14. the 5th term minus the 1st term ... d is 4..
15. get a ... the 5th term... but 5th term is
16. unknown .... 4 times 4 , 16,..
17. 3 times the 5th term ... T_5 ...
18. ... if 10 divided by 3 is T_5.....no..
19. 10th term is not known...
20. 5th term minus 4 and so on....
21. cannot get it... [Stop and give up.]

<u>Figure 14</u>. Protocol of subject B06 illustrating the use of trial-and-error strategy in Item 16

In the protocol analysis, a problem-solving process was classified by noting the order of occurrence of each variable in the protocol and the accompanying reasoning steps according to the examples shown above. Table 15 presents the results of classifying the problem-solving strategies for each problem attempted by the subjects in the task-based interviews.

In Table 15, it can be seen that the working forward strategy was used most often, occupying 64% of the attempted cases. The forward searching strategy came second, being employed in 21% of the cases. Simple arithmetic was used in 9% of the cases. The trial-and-error and the backward reasoning strategies occupied 4% and 2% respectively, the latter being used in 5 cases only. Similar to the observation made in Table 14 on problem representations, some subjects tended to use a particular type of strategy. For example, subject A33 used the working forward strategy most of the time. Subject B17 used both working forward and forward search strategies in high proportion. On the other hand, certain items were associated with the use of strategies of a specific type more often. (Notice that unlike the case of problem representations, a problem can be solved by a variety of strategies.)

The high proportion in using the working forward strategy may be attributed to the fact that the items used in the interviews were mostly familiar problems. At least half of those harder problems were in fact not difficult for the high ability group. So, many subjects could solve them quite readily.

Table 15

Problem-Solving Strategies Employed by Subjects in the Task-based Interviews

Items in task-based interviews	High Ability Good Performance						Low Ability Poor Performance					
	A08	A33	A36	B16	B19	B27	A12	A26	A38	B02	B06	B17
Simple problems												
5. (a)	F 1	F 1	F 1	F 1	F 1	F 1	F 1	F 1	F 1	F 1	F 1	F 1
(b)	F 2	F 1	F 1	F 1	F 2	F 1	F 6	F 1	F 1	F 1	F 1	F 1
(c)	F 1	F 1	F 1	F 1	F 1	F 1	- 7	F 2	FS3	F 1	F 1	FS4
(d)	F 1	F 1	F 1	F 1	F 1	F 2	- 7	F 2	- 5	F 1	F 1	F 1
6.	F 1	F 1	F 1	F 1	F 1	F 1	A	A	A	F 2	FS4	F 1
7.	F 2	F 1	F 1	F 1	F 1	F 1	F 1	T 1	T 1	F 1	FS1	F 1
8. (a)	F 1	F 1	T 1	F 1	F 1	F 1	- 7	- 7	FS5	FS1	F 1	F 2
(b)	F 1	F 1	F 1	F 1	F 1	F 1	A	A	A	F 1	F 2	F 2
(c)	F 1	F 2	F 1	F 1	F 1	F 1	- -	A	A	F 3	F 2	FS2
(d)	F 1	F 1	FS2	F 1	F 1	F 1	- 6	A	FS5	F 3	FS6	FS3
9.	F 2	F 1	F 1	F 1	F 1	F 1	FS3	FS4	FS1	F 1	F 1	F 2
10.	F 2	F 1	F 4	F 1	FS1	F 1	FS4	FS4	T 1	F 1	FS1	FS1
Harder problems												
11. (a)	F 1	F 1	F 1	F 1	F 1	F 1	- 7	FS4	FS4	F 1	FS4	FS1
(b)	F 1	F 1	F 1	FS1	F 1	F 2	- -	F 4	- -	F 1	- -	F 1
12.	F 2	F 1	F 1	F 1	F 2	F 1	A	FS4	T 4	F 2	F 2	B 1
13.	F 2	F 2	F 2	FS1	T 5	FS2	A	A	T 1	FS3	A	FS3
14.	F 1	F 2	A	F 1	F 1	A	A	A	FS2	FS4	FS4	- 5
15.	FS1	F 1	FS2	F 1	FS1	F 1	- 5	- 5	FS4	B 5	FS5	FS5
16.	F 2	FS1	FS1	F 2	FS1	- 5	- 5	- 5		FS5	T 5	FS3
17. (a)	F 2	F 1	F 1	F 1	F 2	F 1	A	A	T 1	F 1	F 1	F 1
(b)	F 2	F 1	F 1	F 1	F 2	F 1	A	A	T 1	F 1	F 2	F 1
18. (a)	F 1	F 1	F 1	F 1	F 1	F 1					F 1	B 1
(b)	F 2	F 1	F 1	F 1	F 3	F 1					FS4	FS3
19.	FS5	FS5	FS1	FS3	FS4	FS1		FS5			FS4	FS5
20. (a)	B 4	B 1	A	FS4	FS4	- 5						
(b)	- -	F 1	A	- -	- -	- -						

F: Working forward
FS: Forward searching
B: Backward reasoning
T: Trial-and-error
A: Work with simple arithmetic

1: all correct
2: correct method with algebraic/arithmetic errors
3: correct method but cannot reach the goal
4: wrong method/approach
5: understand the problem but cannot make any trial
6: misinterpret the problem
7: cannot comprehend the problem
-: inapplicable/give up after reading the problem

Similar to the pilot study, backward reasoning was observed only in a few cases. The underlying reasons might be the same as suggested earlier (pp.79-80). One further possibility for the subjects participating in the present instruction is that owing to the small number of variables involved in this topic, the problem space of any task was in fact very limited. With sufficient practice, subjects could get acquainted to almost all possible combinations of variables which might give solution to other unknowns. So this may be one reason why general reasoning strategies were seldom elicited in the task-based interviews. Either one knew some procedures to solve a problem or one just didn't know anything in that area at all. There was no need to search by backward reasoning in such a restricted problem space. This same reason also explains why the working forward and forward search strategies occupied altogether 85% of the observed cases.

From an alternative perspective, we may say that subjects were actually not accustomed to using search strategies such as backward reasoning or means-ends analysis. Encountering harder problems which did demand some kind of search and reasoning for successful solution, they very often just worked on from the givens in the hope of reaching the goal but without a conscious plan. For instance, Items 13 and 19 are good examples (see Table 15). If these two items were to be solved successfully, backward reasoning would be quite effective.

It is observed that no subjects interviewed used the structural diagrams to analyse the problems in their written solution. The reason may be that the students just took it as a method to teach and did not realize its usage in making a problem situation explicit.

All the subjects interviewed did not spend time to analyse problems before executing some procedures. The first step was most often an algebraic expression or a formula to use. This stage of analysing the problem situation before attempting to solve was particularly crucial for harder problems. For instance, the protocol presented in Figure 10 (p.127) shows that the subject just struggled to substitute some values into a summation formula without even writing down some numbers not divisible by 3 to see what actually were to be added. Figure 17 (p.150, to be discussed later) illustrates another case in that the subject gave up a problem prematurely because he even missed one important information in the problem statement. (He did not consider that the given sum was 1225.)

This absence of the understanding stage may be due to poor training in prescriptive problem-solving procedures. Perhaps, mathematics teachers seldom addressed this issue explicitly. While solving familiar tasks could be mainly schema-driven, novel tasks demanded a more thorough understanding of the problem situation. This observation might explain <u>partly</u> why performance in nonroutine problems of the written test was so poor in both classes. If improvement in general problem-solving is desired, this observation has to be taken into consideration in designing instruction. For this, Polya's heuristics can help.

Differences in the problem-solving performance between students of good and poor performance. When subjects interviewed were divided into two groups in terms of good and poor performance in the problem-solving test, differences in their problem-solving processes could be compared by studying Table 14 (p.132) and Table 15 (p.141). Significant differences were found in the type of problem representations and the type of strategies used. Subjects in the good performance group used the elaborated algebraic representation more often than those in the poor performance group, reflecting their better understanding in the topic. They used mainly the working forward strategy. On the other hand, subjects in the poor performance group might use arithmetic representations which were very seldom employed by good performers. Furthermore, the poor performers used the trial-and-error strategy and arithmetic techniques much more than the good performers.

When details of protocols were considered, we can note the following comparisons by looking at some protocol extracts. For simple problems, students of good performance were more ready to pick out key words from problem statements through which they could immediately recognize typical problems. Consequently, they could apply familiar procedure to work forward to achieve the goal. For example, Figure 15 shows the protocol of subject B27 who belonged to the good performance group. In this problem, it could be observed that the first and the last terms initiated the general term formula which involved n,

Item 6 in Task-based Interviews (A simple problem)

How many terms are there in the following A.P.:
10, 13, 16, ... , 94 ?

Protocol of Subject B27 (Good performance group)

1. How many terms ... in ... A.P.
2. 10 ... 94 ...
3. If there are n terms, 94 equal to
4. 10 plus (n - 1) times ... common difference
5. is 3. [Write down an equation in n.]
6. 84 is equal to 3 times (n - 1).
7. Both sides divided by 3.
8. [Do mental calculations.]
9. 28 is n-1 ...
10. There are 29 terms.

Written Solution

$94 = 10 + (n - 1).3$	(lines 3-5)
$84 = 3 (n - 1)$	(line 6)
$28 = n - 1$	(line 9)
$n = 29$	(line 10)

Tree Diagram

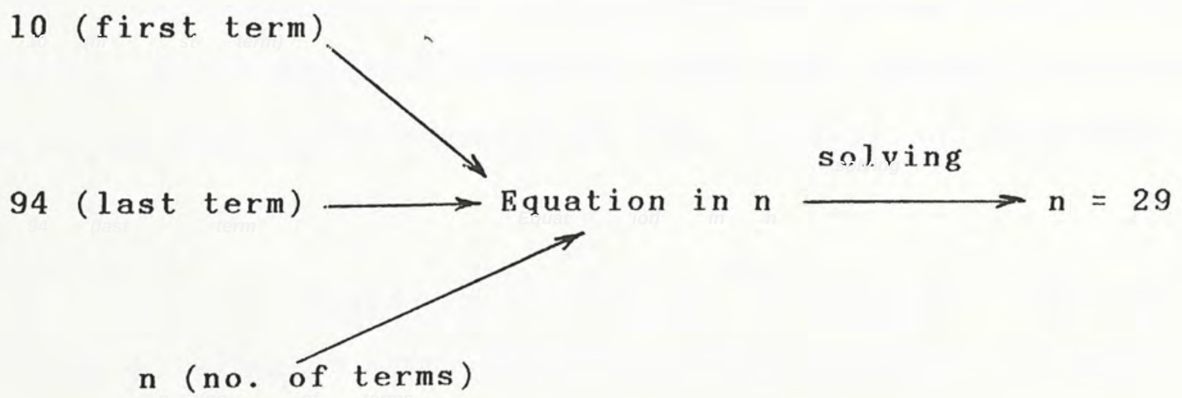


Figure 15. Protocol of subject B27 in solving a simple problem (Item 6).

the number of terms (lines 2-4). Minor details in the problem such as the value of the common difference did not come to the subject's mind until execution of procedures (line 4). The representation using the standard algebraic notation by the subject was very effective. Thus, the tree diagram shows that the process proceeded smoothly. The overall structure of the solution was simple, goal-directed and consistent with the knowledge required for solving this problem.

Figure 16 shows another subject B02 solving the same problem. This subject belonged to the poor performance group. Although a relevant procedure was recalled (line 2), the subject did not get a correct answer as the exact meaning of the division was not correctly interpreted with respect to the problem elements (lines 4-5). While he worked forward quite promptly on reading the problem statement and no search was exhibited in the process, very much like that of good performance group, his failure could be attributed to an incorrect problem representation which in turn was related to his knowledge base as discussed in the section on knowledge structures.

With simple problems, a variety of problem-solving strategies was observed among students of poor performance though most of them could be classified as working forward (refer to the top right quadrant of Table 15). However, their performance was qualitatively poorer compared with that of the good performance group in terms of problem representation and the structure of solution, Item 6 in Task-based Interviews (A simple problem)

How many terms are there in the following A.P.:

10, 13, 16, ... , 94 ?

Protocol of Subject B02 (Poor performance group)

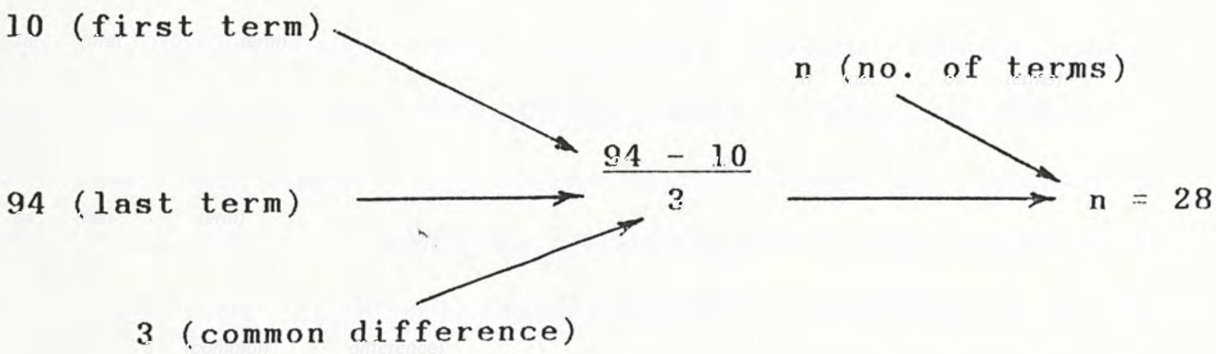
1. [Read the question.]
2. [Pause.] 94 minus 10 divided by 3 ...
3. n is equal to ...
4. [Work with the calculator.]
5. 28 ....
6. There are 28 terms.

Written Solution

$\frac{94 - 10}{3} = n$ (line 2)

$n = 28$ (line 5)

Tree Diagram



Note. The answer was wrong here. The subject had not related correctly the number of terms with the number of common differences in between.

Figure 16. Protocol of subject B02 in solving a simple problem (Item 6).

the difference in the latter being particularly significant when forward search was made without a clear goal.

With harder problems, subjects in the good performance group still worked with forward strategies (refer to Table 15). This time, either they might recognize that a problem was a combination of several basic tasks or they just started with certain subgoals which they could solve or they believed to be useful sooner or later. An example of this strategy can be observed in the protocol exhibited in Figure 5 (p.104) discussed previously. In this example, Item 15 was in fact broken into two subproblems, the first one being to find the number of terms and the second being to find the common difference. Among good performers, these working forward/forward searching strategies were observed to be rather frequent and were actually found to be quite successful in solving the given tasks (refer to Table 15 where the code number indicates the success or failure to solve). However, it must be stressed that the success of problem-solving in these cases should be attributed to the problem representation generated by the subject and not so much to the direction of working. As a matter of fact, the latter could be regarded as an overt behaviour manifesting the quality of problem representation.

On the other hand, when presented with harder problems, subjects from the poor performance group did not switch strategies used in simpler tasks (compare the upper and the lower parts of the right side of Table 15). They again applied any procedures they could think of which might start from the givens in the problem. Meaning of newly generated values from the formulas might not be correctly interpreted. But they just expected that these values could somehow lead to the goal. Of course, in such harder problems, it was quite improbable that an answer could be attained by chance. Thus, success was rare. This is also indicated by the code numbers in Table 15.

Basically, subjects in the poor performance group just didn't seek any understanding of the problem before they tried. Procedures were applied only to those variables they were familiar with and other data might be entirely neglected in extreme cases. This approach certainly meant failure when conventional tasks were all composed of minimal given conditions. Figure 17 is an example of how a subject of poor performance decided to give up a problem because he thought that two unknowns were closely related but neither one could be obtained first. If he could make sure how many givens there were in the problem, he might break the loop that caused the abandonment (lines 4-7 form a loop). (He did use in other items the summation formula that was essential to solve n here.) Thus, it may be said that the problem representation generated by subjects of poor performance might be incomplete in some cases. This is also an important reason for their poor performance.

Item 15 in Task-based Interviews (A harder problem)

An A.P. 42, ... , 133 has a sum of 1225,
i.e.

$$42 + \dots + 133 = 1225.$$

Find the common difference of this A.P.

Protocol of Subject B02 (Poor performance group)

1. [Read the question.] ... Use 42.
2. Use 42 and 133 to find the common difference.
3. ... But how to find ...
4. To find the common difference first? ...
5. No. Find the number of terms first. [Pause.]
6. ... But to find the number of terms requires
7. the value of d ! ...
8. [Stop and give up.]

Written Solution Nil

Tree Diagram

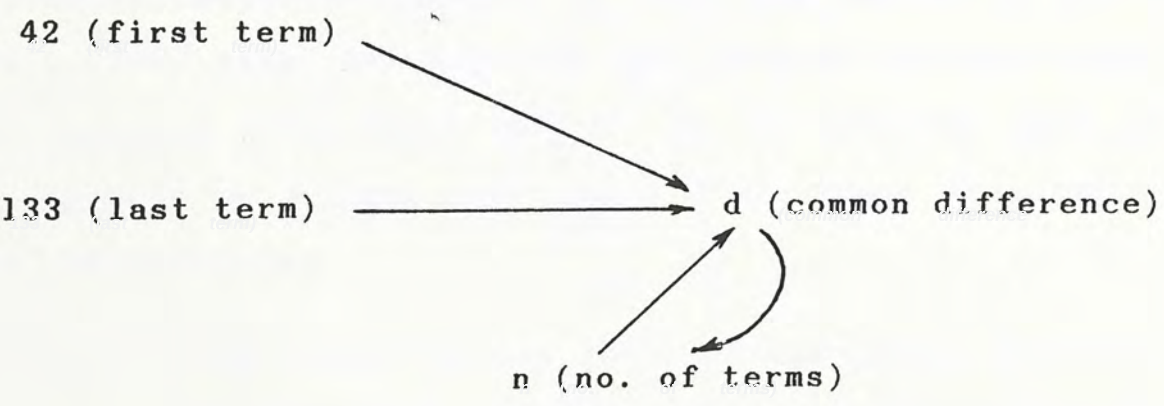


Figure 17. Protocol of subject B02 in solving a harder problem (Item 15).

In sum, it was found that the mathematical reasoning ability of the students tended to correlate with the problem-solving performance, manifesting in terms of the type of strategies used and the quality of problem representations. Subjects of good performance used more elaborated problem representations, reflecting better understanding of the topic and mathematical reasoning ability. With harder problems, they could break them down into solvable subproblems. This also resulted in a clear structure of their solution. These observations were primarily in line with those of expert-novice studies.

Differences in the problem-solving performance between students of the control group and the experimental group. When interviewed subjects were divided into groups according to the instruction received, i.e. group A and group B, the differences in their problem-solving processes could be compared by studying Tables 14 and 15 again. This time, subjects of code numbers started with A are compared with those with B. It is observed that the differences were less noticeable than those observed in the last section.

Subjects in the good performance division exhibited very similar problem-solving protocols and solutions in both groups. Tables 14 and 15 demonstrate this similiarity very clearly. As far as the collected protocols could show, there were no significant differences observed in their problem representation as well as the solution strategies.

Subjects from the lower performance division showed differences in the strategies employed between the two groups. In Table 15, it was observed that those from group B receiving instruction with the use of structural diagrams employed forward working or forward searching strategies more often than their counterparts in group A. The nature of these strategies was more knowledge-based. Of course, the use of strategies was not equivalent to successful problem-solving. But that indicated less able subjects in group B had mastered some formulas or procedures which they could continuously rely upon in tackling problems. By contrast, the three subjects from group A switched strategies from problem to problem.

Similarly, Table 14 shows that subjects of the lower performance division in group B tended to use the formal algebraic representation more often than those in group A. This difference reflected the differences observed in their inferred knowledge structures. Algebraic formulation was better acquired by subjects in group B. Despite this difference, it should be noted that their problem representation, irrespective of types, were very often incomplete or inappropriate particularly with the harder problems. As pointed out previously, the two groups did not differ in their understanding of the concepts. So it follows that their problem representations suffered similiar faults. These findings would be consistent with those on knowledge structure differences.

In view of these differences observed between the two groups, we can conclude that the use of structural diagrams did have some effects on the problem-solving strategies and the type of problem representation, the lower ability subjects in the treatment group being more prone to use forward strategies and the formal algebraic representation. However, this did not have much effects on the quality of the problem representation. Based on the above qualitative comparisons, a weak interaction effect of the instructional method and the general mathematical ability level on the use of problem-solving strategies and on the type of problem representations chosen was thus observed. This might be due to the interaction effect on the underlying knowledge structure as we have found previously.

Nevertheless, since using a particular problem-solving strategy does not guarantee good performance, this positive effect on using the forward strategies does not seem to be much desirable. Rather, these findings revealed the significant role of understanding in acquiring knowledge. That is to say, the instructional goal should aim at how concepts may be elaborated and represented by useful models, how relationships between knowledge units can be fostered, and how knowledge units can be flexibly organized to facilitate problem-solving. As to these, structural diagrams do not seem to contribute much as the present study indicates.

Relationships between the Inferred Knowledge Structure and the Problem-Solving Performance

The results and discussions so far should make it clear that although knowledge structures were inferred from performance in some specific tasks, inferred knowledge structure and problem-solving performance could still be distinguished. The construction of knowledge structure introduced hypotheses concerning the organization and also took into account the subject's performance in a variety of problems.

As the process of analysing protocols showed, consistent good problem-solving performance would support more hypothesized linkages between knowledge units. More available procedures were inferred. Elegant problem representation evidenced hypothesized efficient or elaborated representation of knowledge units. This was what had actually been done during the stage of analysing raw data. By contrast, inconsistent problem-solving performance might indicate conceptual misunderstanding. Contradictory responses led to the conclusion that the subject might not relate some knowledge units together so that inconsistency could persist. What we were getting at was some common core underlying each of the task performance observed.

Of course, the results obtained demonstrated that subjects with an inferred knowledge structure rich in relationships and with available procedures for problem-solving certainly performed well in problem solving tasks.

This conclusion was in agreement with other similar studies. However, knowledge structure as an inferred construct finds its strength in the power of explanation and prediction. More specifically, in the present study, the representations built up for the knowledge structure of the subjects interviewed, if valid, could be applied in predicting their individual performance when given some other tasks in this topic. Until verification of predictions could be made with high certainty, the relationships between the inferred knowledge structure and problem-solving performance could be firmly established. In this respect, the models obtained here were very primitive and other knowledge components that would be likely to interact with knowledge of arithmetic progressions should have to be considered too.

CHAPTER VI CONCLUSIONS, IMPLICATIONS AND RECOMMENDATIONS

Conclusions

In the present study, the effects of using structural diagrams in mathematics instruction were investigated with regard to the acquisition of knowledge structure and problem-solving performance. Positive significant effects were hypothesized based on the possible functions of structural diagrams and the theory of knowledge-based problem-solving. But not all hypothesized effects were observed.

The effect of the use of structural diagrams on the problem-solving test scores.

1. There was significant difference in the routine problem score between the control and the experimental groups, favouring the experimental group.
2. There was no significant difference in the nonroutine problem score between the two groups.
3. There was significant difference in the total problem-solving score between the two groups, favouring the experimental group.
4. There were no significant two-way interaction effects of the instructional method and the mathematical reasoning ability level on all these three problem-solving test scores.

The effect of the use of structural diagrams on the acquisition of knowledge structure.

1. The use of structural diagrams had positive effects on the acquisition of knowledge structures. But, effects were only observed on the acquisition of the standard formulas particularly in the lower ability division. There was no significant difference in the conceptual understanding and the overall organization of the knowledge structure.
2. There was a marginal interaction effect of the instructional method and the mathematical ability level on the acquisition of knowledge structures, the low ability division in the experimental group benefiting more from the treatment.

The effect of the use of structural diagrams on the problem-solving performance.

1. More frequent use of working forward or forward searching strategies was observed in the experimental group, particularly in the lower ability division.
2. Higher tendency to use the formal algebraic representation was observed in the experimental group, particularly in the lower ability division.
3. There was no significant effect on the quality of the problem representation.
4. A marginal interaction effect of the instructional method and the mathematical ability level on the problem-solving performance was observed, favouring the lower ability division in the experimental group.

Comparison of inferred knowledge structures and problem-solving performance between the good and the poor performance groups.

1. Subjects of the good performance group possessed knowledge structures significantly different from those of the poor performance group in terms of conceptual understanding and knowledge of procedural rules. Their knowledge structures contained more relationships built up between knowledge items and resembled more closely to the standard structure.
2. Subjects in the good performance group used the elaborated algebraic representation more often than those in the poor performance group. They used mainly the working forward strategy. On the other hand, subjects in the poor performance group might use arithmetic representations which were very seldom employed by good performers. Furthermore, the poor performers used the trial-and-error strategy and arithmetic techniques much more than the good performers.

Results concerning the effects of the use of structural diagrams on the problem-solving test scores, the acquisition of knowledge structures, and problem-solving performance were consistent with each other. Comparisons between the good and the poor performers were in general agreement with those of expert-novice studies. But it was observed that backward reasoning and means-ends analysis were almost absent from the strategies used by the subjects interviewed, contrary to the well-documented finding that novices tended to use general search strategies such as means-ends analysis.

Implications

Based on the results of the experimental instructional programme, it can be suggested that structural diagrams can be accepted as a convenient means of demonstrating problem structures and explaining the relationships between knowledge units. It is particularly effective for identifying and representing basic problems. It also facilitates the acquisition of basic problem-solving schemata. Nevertheless, this experiment also indicated its effects were quite limited in scope. Structural diagrams failed to help students in conceptual understanding and in handling novel problems. Further means have to be devised for more effective instruction of conceptual knowledge.

The findings in knowledge structure and problem-solving process again ascertain the role of problem representation in successful problem-solving. This generation of internal representations requires understanding which very often depends on specific knowledge in certain area. Furthermore, as observed in task-based interviews, subjects were not familiar with systematic problem-solving procedures, say, the very first stage of analysing the problem. The subjects participating were very weak at strategies of reasoning and searching. This suggests that the students might be receiving very poor training in the general procedures of problem-solving. In their mathematics lessons, they might have acquired only a collection of mathematical facts and formulas. To sum up, we can say that instruction for successful problem-solving in mathematics has to foster understanding as well as to introduce general procedures. The metacognitive skills have to be taught explicitly.

Some mathematics teachers may tend to emphasize only the learning and retention of formulas and rules instead of understanding. They seldom demonstrate the consistency of different approaches to the same problem. Alternative solutions are usually not encouraged. These practices may have detrimental effects on the learners. Understanding necessitates the cultivation of a more active role on the part of the students in relating different pieces of mathematical knowledge and seeking for internal coherence. However, this metacognitive aspect of recognizing logical consistency as an important criterion for the validity of mathematical knowledge belongs to the more general realm of mathematics education. It cannot be addressed by the present short instructional programme.

Task-based interviews conducted in the study have revealed many interesting points on students' understanding of the topic of "arithmetic progression". These might not be known to mathematics teachers who primarily used conventional methods to assess students. However, findings of this sort will be very useful when we desire to base instructional design on a firm empirical ground. It is expected that more detailed empirical information should be similarly explored in other areas of the mathematics curriculum.

Furthermore, in the present instructional programme, a concrete representation for an arithmetic progression was introduced. Some students could use it in a very flexible manner so that problems could be solved effectively and with understanding. The success of this attempt leads us to speculate if it is possible to devise such "mental models" to help with understanding and manipulation in other topics of mathematics.

Recommendations

According to what were found in the present study, recommendations are made in two aspects, instruction and future research.

Mathematics instruction.

1. It is recommended that teachers may attempt to use diagnostic tests or items similar to those in the present study to get empirical information on the knowledge structures possessed by the students. This step is necessary for an effective course in mathematics instruction. The data collected will be useful for planning instruction. This can be considered as a practical attempt to link up the theory of instruction and the theory of learning.

2. Worked examples should be demonstrated with the aim of exposing the underlying problem structure as well as more understanding of the mathematical entities involved. There are a variety of ways. Structural diagrams can be an useful aid. The acquisition of specific

schemata can be thus facilitated. Teachers may encourage students to solve problems in as many ways as they can so that more linkages between knowledge units would be constructed or activated.

3. Many teachers think that worked examples are sufficient to illustrate the methods involved and students can develop strategies on their own. However, it may not be the case for students of low ability. General strategies are recommended to be explicitly discussed using worked examples so that students can gain some flexibility to use general procedures in handling problems.

Further research.

1. As human minds excel in knowledge acquisition and novel utilization, the present trend of research in modelling human knowledge and its acquisition process will certainly continue. As earlier reviewed, application of this cognitive science approach to some specific topic of school mathematics has not yet played an important role in mathematics education research. Studies of this sort are recommended. Such research can have both theoretical and practical significance.

2. The present study failed in investigating the process of constructing problem representations. More reliable verbalization data have to be obtained before a better understanding of the exact process can be attained. As this is a crucial step in solving problems, effective methods for collecting data have to be devised. While

precise change in the problem statements can be easily manipulated, it is not as easy to control the knowledge base of the subject. So other knowledge components likely to interact with the process have to be taken into consideration too.

3. In order to control the effect of instruction more precisely in investigating the learning and knowledge acquisition processes, it is recommended that the corresponding knowledge representation should be changed as precise and detail as possible and such learning research programmes could be implemented in micro-computers. The interactive environment provided by the computers makes it possible for manipulation as precise as we desire.

4. Concerning the meaning and validity of the construct "knowledge structure", a suggestion may be considered. Models of inferred knowledge structure can be further used for the purpose of predicting possible responses upon some tasks specifically designed. If a model of knowledge structure for a particular individual is successful in making predictions that can reflect the general performance including the actual mistakes or errors made by the subject, the model can gain more validity. This dynamic aspect of the knowledge structure would be very promising for future research.

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Appendix A

Mathematics Achievement Test

Time allowed: 1 hour

Notes :

1. This paper has 40 multiple choice questions.
2. Not all diagrams are drawn to scale.
3. Arrows in geometrical figures indicate parallel lines.

注意：

㈠本卷有40題多項選擇題。
㈡部份附圖不依比例繪畫。
㈢幾何圖形中箭號代表平行線。

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(1) If N is a natural number, which of the following must be an <u>odd</u> number ?

A. $N+1$

B. $2N+1$

C. $3N$

D. N^2

E. N^2+1

(1) 若N是一個自然數，則下列那一項必是<u>奇數</u>？

A. $N+1$

B. $2N+1$

C. $3N$

D. N^2

E. N^2+1

---

(2) Which of the following is the best approximation of $\dfrac{2.01^2+7.95^2}{(0.1875\times 10)^2}$?

A. 12

B. 15

C. 17

D. 25

E. 50

(2) 下列何者是 $\dfrac{2.01^2+7.95^2}{(0.1875\times 10)^2}$ 的最佳近似值？

A. 12

B. 15

C. 17

D. 25

E. 50

---

(3) Among 800 students, 30% chose Computer Studies. In this group, there were 40 more boys than girls. How many girls chose Computer Studies ?

A. 100

B. 140

C. 200

D. 240

E. 280

(3) 學生800人，其中30%選修電腦科。在這組學生中，男生比女生多40人。問有多少女生選修電腦科？

A. 100

B. 140

C. 200

D. 240

E. 280

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(4) John is a salesman. He gets a weekly salary of \$1 000. In addition, he receives a commission of $\frac{1}{2}\%$ of the amount of his weekly sales. Last week, his total income was \$1 250. What was the amount of his sales ?

A. \$5 000
B. \$25 000
C. \$50 000
D. \$200 000
E. \$250 000

(4) 約翰是一位售貨員，除週薪 \$1 000 外，他更可從他經手的一週營業額中，抽取 $\frac{1}{2}\%$ 作爲酬金。上週，他的總收入共 \$1 250，問他的營業額是多少？

A. \$5 000
B. \$25 000
C. \$50 000
D. \$200 000
E. \$250 000

(5) In a children art club, 60% of the members are boys. If the number of boys exceeds the number of girls by 18, how many girls are there in this club ?

A. 12
B. 30
C. 36
D. 45
E. 54

(5) 在某兒童繪畫班中，男童佔 60%。若男童比女童多 18 人，問女童共有幾人？

A. 12
B. 30
C. 36
D. 45
E. 54

(6) A man walks 2 km in the direction N45°E from P to Q. He then walks another 2 km in the direction S45°E from Q to R. What is the compass bearing of R from P ?

A. N45°W
B. S45°W
C. Due West
D. Due North
E. Due East

(6) 某人從 P 向 N45°E 走了 2 km 至 Q。然後再由 Q 轉向 S45°E 走 2 km 至 R。問 R 對於 P 的方位是甚麼？

A. N45°W
B. S45°W
C. 正西
D. 正北
E. 正東

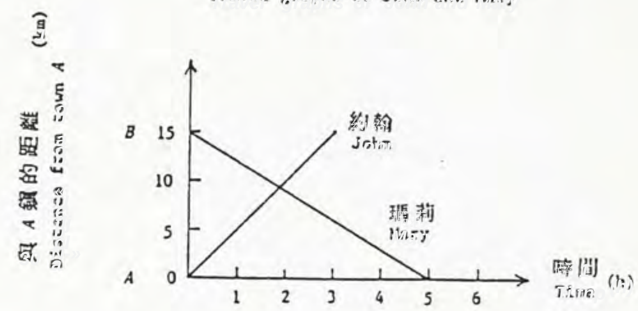
(7) In a 60 km bicycle race, the fastest finished the race in 4 hours while the slowest finished in 5 hours. Hence the speeds of other competitors must lie between

A. 4 km/h and 5 km/h
B. 10 km/h and 12 km/h
C. 12 km/h and 15 km/h
D. 15 km/h and 20 km/h
E. 40 km/h and 50 km/h

(7) 在一項 60 km 單車賽中，最快抵達終點的用了 4 小時，最慢的用了 5 小時。由此可知，其他參賽者的速率一定在

A. 4 km/h 與 5 km/h 之間
B. 10 km/h 與 12 km/h 之間
C. 12 km/h 與 15 km/h 之間
D. 15 km/h 與 20 km/h 之間
E. 40 km/h 與 50 km/h 之間

約翰及瑪莉的旅程圖
Travel graphs of John and Mary

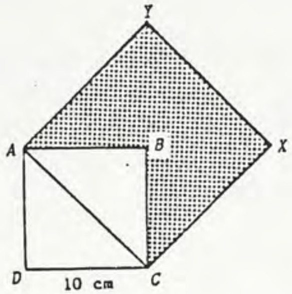


(8) The above diagram shows the travel graphs of John and Mary. John travelled from town A to town B while Mary travelled from B to A. By how much was John's speed greater than Mary's ?

A. 0 km/h
B. 2 km/h
C. 3 km/h
D. 5 km/h
E. 10 km/h

(8) 上圖所示爲約翰及瑪莉的旅程圖。約翰從 A 鎮往 B 鎮，而瑪莉則由 B 鎮往 A 鎮。問約翰的速率比瑪莉的快多少？

A. 0 km/h
B. 2 km/h
C. 3 km/h
D. 5 km/h
E. 10 km/h

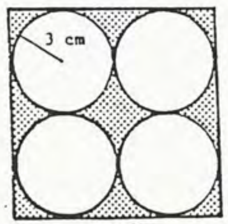


(9) The above figure shows two squares $ABCD$ and $ACXY$. If $CD = 10$ cm, what is the area of the shaded part ?

A. 100 cm^2
B. 150 cm^2
C. 200 cm^2
D. 300 cm^2
E. 350 cm^2

(9) 上圖所示爲兩個正方形 $ABCD$ 及 $ACXY$。若 $CD = 10$ cm，間有陰影部份面積是多少？

A. 100 cm^2
B. 150 cm^2
C. 200 cm^2
D. 300 cm^2
E. 350 cm^2

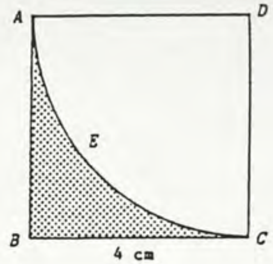


(10) The above figure shows four equal circles, each of radius 3 cm, inscribed in a square and touching two others. The total area of the shaded parts in cm^2 is

A. $36(4 - \pi)$
B. $36(2 - \pi)$
C. $36(1 - \pi)$
D. $24(6 - \pi)$
E. $12(3 - 2\pi)$

(10) 上圖所示爲四個等圓，每個圓的半徑都是 3 cm，內切於一正方形，且與其他二圓相切。有陰影部份的總面積（以 cm^2 表示）是

A. $36(4 - \pi)$
B. $36(2 - \pi)$
C. $36(1 - \pi)$
D. $24(6 - \pi)$
E. $12(3 - 2\pi)$

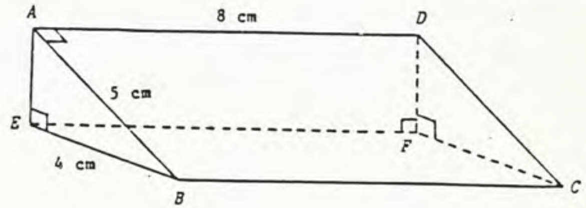


(11) In the figure above, $ABCD$ is a square of length 4 cm. $DAEC$ is a sector. The area of the shaded part in cm^2 is

A. $16 - \pi$
B. $2(8 - \pi)$
C. $4(4 - \pi)$
D. $8(2 - \pi)$
E. $16(1 - \pi)$

(11) 在上圖中，$ABCD$ 是一個邊長 4 cm 的正方形，$DAEC$ 是一個扇形。有陰影部份的面積（以 cm^2 表示）是

A. $16 - \pi$
B. $2(8 - \pi)$
C. $4(4 - \pi)$
D. $8(2 - \pi)$
E. $16(1 - \pi)$



(12) The above figure shows a prism with triangular cross-section. If $AD = 8$ cm, $AB = 5$ cm and $BE = 4$ cm, then the volume of the prism in cm^3 is

A. 16
B. 32
C. 48
D. 96
E. 160

(12) 上圖表示一角柱體，橫切面是三角形。若 $AD = 8$ cm，$AB = 5$ cm 及 $BE = 4$ cm，則該角柱體的體積（以 cm^3 表示）是

A. 16
B. 32
C. 48
D. 96
E. 160

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(13) The volume V of a sphere can be found by the formula $V=\frac{4}{3}\pi R^3$ where R is its radius. If a sphere has the same volume as a cube of length 2 cm, then its radius is

A. $\sqrt[3]{\frac{32\pi}{3}}$ cm

B. $\sqrt[3]{\frac{6}{\pi}}$ cm

C. $\sqrt[3]{\frac{32}{3\pi}}$ cm

D. $\sqrt[3]{6\pi}$ cm

E. $\sqrt[3]{\frac{2}{3\pi}}$ cm

(13) 一個球體的體積 V 可用 $V=\frac{4}{3}\pi R^3$ 這公式求得，其中 R 是它的半徑。若某球體與一個邊長 2 cm的正方體體積相等，則它的半徑是

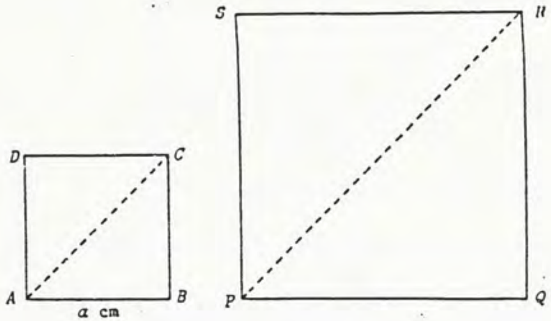
A. $\sqrt[3]{\frac{32\pi}{3}}$ cm

B. $\sqrt[3]{\frac{6}{\pi}}$ cm

C. $\sqrt[3]{\frac{32}{3\pi}}$ cm

D. $\sqrt[3]{6\pi}$ cm

E. $\sqrt[3]{\frac{2}{3\pi}}$ cm



(14) The above diagram shows two squares $ABCD$ and $PQRS$. If $AB=a$ cm and $PR=2AC$, what is the area of $PQRS$ in cm^2 ?

A. $2a^2$

B. $4a^2$

C. $\sqrt{8}a^2$

D. $8a^2$

E. $8a$

(14) 上圖所示爲兩個正方形 $ABCD$ 及 $PQRS$。若 $AB=a$ cm 及 $PR=2AC$，問 $PQRS$ 的面積（以 cm^2 表示）是多少？

A. $2a^2$

B. $4a^2$

C. $\sqrt{8}a^2$

D. $8a^2$

E. $8a$

(15) If $a=\frac{3}{4}b$ and $b=\frac{8}{9}c$, then $a:b:c=$

A. 3 : 4 : 8

B. 3 : 4 : 9

C. 3 : 8 : 9

D. 6 : 8 : 9

E. 9 : 8 : 6

(15) 若 $a=\frac{3}{4}b$ 及 $b=\frac{8}{9}c$ ，則 $a:b:c=$

A. 3 : 4 : 8

B. 3 : 4 : 9

C. 3 : 8 : 9

D. 6 : 8 : 9

E. 9 : 8 : 6

(16) If $y=2x^2-3x+4$ and $x=-2$, then $y=$

A. 2

B. 6

C. 16

D. 18

E. -10

(16) 若 $y=2x^2-3x+4$ 及 $x=-2$ ，則 $y=$

A. 2

B. 6

C. 16

D. 18

E. -10

(17) If $(x+y)^2=a$ and $xy=b$, then $x^2+y^2=$

A. ab

B. $a-b$

C. $a+b$

D. $a-2b$

E. $a+2b$

(17) 若 $(x+y)^2=a$ 及 $xy=b$，則 $x^2+y^2=$

A. ab

B. $a-b$

C. $a+b$

D. $a-2b$

E. $a+2b$

(18) If $4x^2+2kx+9=(2x-3)^2$ is an identity, then $k=$

A. 12

B. -12

C. 0

D. 6

E. -6

(18) 若 $4x^2+2kx+9=(2x-3)^2$ 是一恆等式，則 $k=$

A. 12

B. -12

C. 0

D. 6

E. -6

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(19) Given $a = \frac{bc}{d+e}$. If $a = 2$, $b = 3$, $c = 4$ and $d = 5$, what is the value of e ?

A. $-\frac{3}{2}$

B. $\frac{6}{5}$

C. $\frac{17}{5}$

D. 12

E. 1

(19) 已知 $a = \frac{bc}{d+e}$ 。若 $a = 2$ ，$b = 3$ ，$c = 4$ 及 $d = 5$ ，問 e 的值是多少？

A. $-\frac{3}{2}$

B. $\frac{6}{5}$

C. $\frac{17}{5}$

D. 12

E. 1

(20) If x is a positive number, and $x^{-2} = 4$, then $x =$

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{1}{16}$

D. 2

E. 6

(20) 若 x 是一個正數，且 $x^{-2} = 4$ ，則 $x =$

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{1}{16}$

D. 2

E. 6

(21) When 8 is subtracted from a number x, the result is $\frac{2}{3}x$. What is the value of x ?

A. $5\frac{1}{3}$

B. $7\frac{1}{3}$

C. $8\frac{2}{3}$

D. 12

E. 24

(21) 從一個數 x 減去 8 ，結果是 $\frac{2}{3}x$ 。問 x 的值是多少？

A. $5\frac{1}{3}$

B. $7\frac{1}{3}$

C. $8\frac{2}{3}$

D. 12

E. 24

(22) $9a^2 - 25b^2 =$

A. $(9a + 25b)(a - b)$

B. $(9a - 25b)(a + b)$

C. $(9a - b)(a + 25b)$

D. $(3a + 5b)(3a - 5b)$

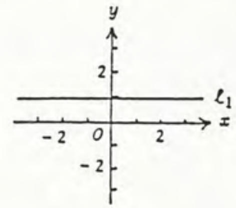
E. $(3a - 5b)^2$

x	-2	0	2
y	3	1	-1

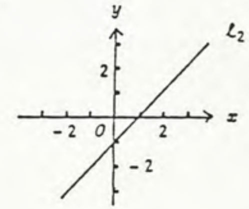
(23) Which of the following graphs represents the relation of x and y shown in the above table ?

(23) 下列那一個圖像代表上表中 x 與 y 的關係？

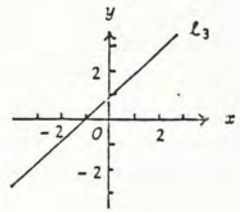
A.



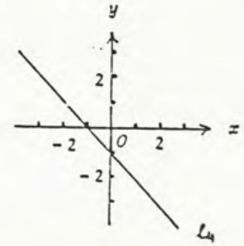
B.



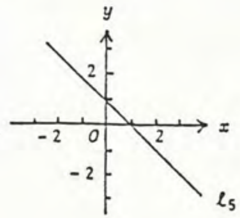
C.



D.



E.



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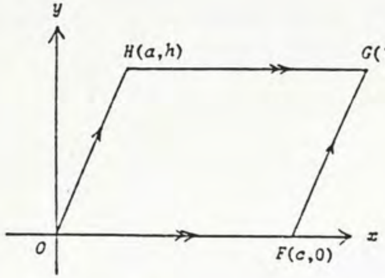


(24) In the figure above, O is the origin, and $OFGH$ is a parallelogram. The coordinates of G are

A. (c,h)
B. (h,c)
C. $(h,c+a)$
D. $(c+a,h)$
E. $(c+h,a)$

(24) 在上圖中，O 是原點，$OFGH$ 是一個平行四邊形。G 的座標是

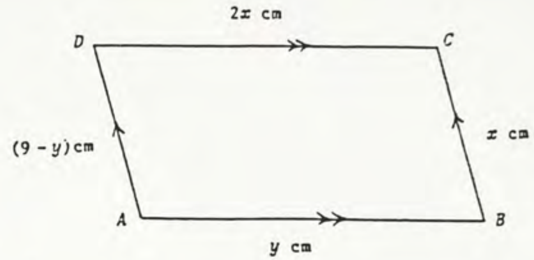
A. (c,h)
B. (h,c)
C. $(h,c+a)$
D. $(c+a,h)$
E. $(c+h,a)$

(25) A bag contains some one-dollar coins and some two-dollar coins. The total number of coins is 51 and the coins amount to \$66. How many two-dollar coins are there ?

A. 15
B. 17
C. 22
D. 25
E. 33

(25) 袋內有一元及二元硬幣共51個，總值 \$66。問二元硬幣有幾個？

A. 15
B. 17
C. 22
D. 25
E. 33

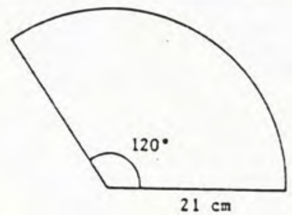


(26) The figure above shows a parallelogram $ABCD$. What is the length of BC ?

A. 2 cm
B. 3 cm
C. 5 cm
D. 7 cm
E. 9 cm

(26) 上圖所示爲一平行四邊形 $ABCD$。問 BC 的長度是多少？

A. 2 cm
B. 3 cm
C. 5 cm
D. 7 cm
E. 9 cm



(27) The above figure shows a sector of radius 21 cm. If π is taken as $\frac{22}{7}$, then the area of the sector in cm^2 is

A. 37
B. 44
C. 47
D. 307
E. 462

(27) 上圖所示爲一扇形，半徑是 21 cm。若以 $\frac{22}{7}$ 爲 π 之值，則該扇形之面積（以 cm^2 表示）是

A. 37
B. 44
C. 47
D. 307
E. 462

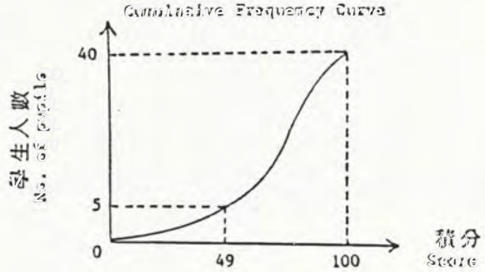
(28) The mean of five numbers 1, a, b, c and 14 is 6. What is the mean of a, b and c?

A. 6
B. 5
C. 4
D. 3
E. cannot be found

(28) 1、a、b、c 及 14 這五個數的平均數是 6。問 a、b 及 c 的平均數是多少？

A. 6
B. 5
C. 4
D. 3
E. 無法求得

累積頻數曲線
Cumulative Frequency Curve



(29) The above graph is a cumulative frequency curve of the scores in a certain mathematics test. If 50 is the passing mark, which of the following statements is/are true?

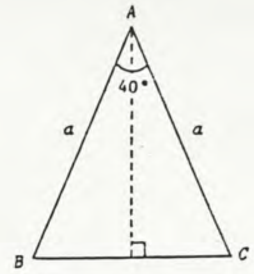
(I) Only 5 pupils passed.
(II) 35 pupils passed.
(III) 40 pupils participated in the test.

A. (I) only
B. (II) only
C. (I) and (II) only
D. (II) and (III) only
E. (I) and (III) only

(29) 上圖為某次數學測驗積分的累積頻數曲線。若及格分數為 50，問下列各項，何者為真？

(I) 只有 5 個學生及格。
(II) 有 35 個學生及格。
(III) 有 40 個學生參加測驗。

A. 只有 (I)
B. 只有 (II)
C. 只有 (I) 及 (II)
D. 只有 (II) 及 (III)
E. 只有 (I) 及 (III)

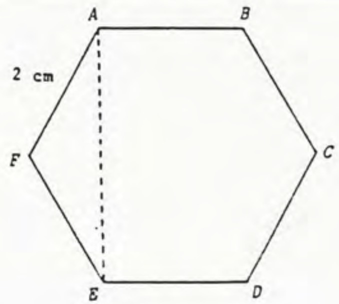


(30) In the above triangle, $AB = AC = a$ and $\angle BAC = 40°$. What is the length of BC?

A. $a \sin 40°$
B. $a \cos 40°$
C. $a \tan 40°$
D. $2a \sin 20°$
E. $2a \cos 20°$

(30) 在上圖的三角形中，$AB = AC = a$ 及 $\angle BAC = 40°$。問 BC 的長度是多少？

A. $a \sin 40°$
B. $a \cos 40°$
C. $a \tan 40°$
D. $2a \sin 20°$
E. $2a \cos 20°$



(31) In the figure above, $ABCDEF$ is a regular hexagon of side 2 cm. AE =

A. $2\sqrt{3}$ cm
B. $2\sqrt{2}$ cm
C. $\sqrt{3}$ cm
D. 3 cm
E. 2 cm

(31) 在上圖中，$ABCDEF$ 是一個正六邊形，邊長 2 cm。AE =

A. $2\sqrt{3}$ cm
B. $2\sqrt{2}$ cm
C. $\sqrt{3}$ cm
D. 3 cm
E. 2 cm

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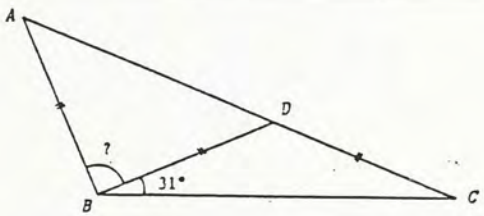
(32) The above pie chart shows the regions where the F.3 students of a certain school lived. If 50 students lived on Hong Kong Island, how many F.3 students were there in that school ?

A. 100
B. 150
C. 180
D. 260
E. 360

(32) 上圖所示為某校中三學生居住地區的圓瓣圖。若住在香港島的佔 50 人，問該校中三學生有幾人？

A. 100
B. 150
C. 180
D. 260
E. 360

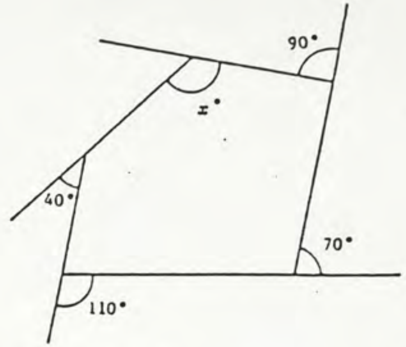


(33) In the figure above, ADC is a straight line and $AB = BD = CD$. If $\angle CBD = 31°$, then $\angle ABD =$

A. 56°
B. 59°
C. 62°
D. 64°
E. 76°

(33) 在上圖中，ADC 是一直線，且 $AB = BD = CD$ 。若 $\angle CBD = 31°$ ，則 $\angle ABD =$

A. 56°
B. 59°
C. 62°
D. 64°
E. 76°

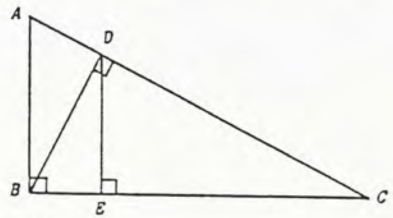


(34) In the figure above, $x =$

A. 110
B. 120
C. 130
D. 140
E. 150

(34) 在上圖中， $x =$

A. 110
B. 120
C. 130
D. 140
E. 150



(35) In the figure above, which of the following triangles is/are similar to ΔABC ?

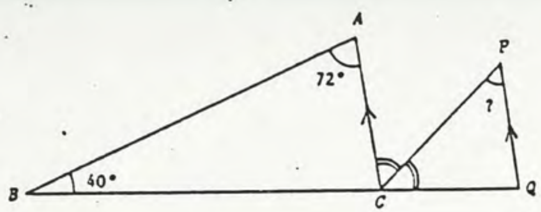
(I) ΔDEC
(II) ΔBDC
(III) ΔADB

A. (I) only
B. (II) only
C. (III) only
D. (I) and (II) only
E. (I), (II) and (III)

(35) 在上圖中，下列各三角形何者與 ΔABC 相似？

(I) ΔDEC
(II) ΔBDC
(III) ΔADB

A. 只有(I)
B. 只有(II)
C. 只有(III)
D. 只有(I)及(II)
E. (I)，(II)及(III)

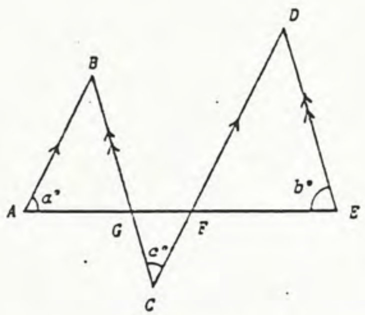


(36) In the figure above, BCQ is a straight line. CP bisects $\angle ACQ$. If $CA \parallel QP$, then $\angle CPQ$ =

A. 40°
B. 45°
C. 56°
D. 68°
E. 72°

(36) 在上圖中，BCQ 爲一直線。CP 平分$\angle ACQ$。若 $CA \parallel QP$，則$\angle CPQ$ =

A. 40°
B. 45°
C. 56°
D. 68°
E. 72°

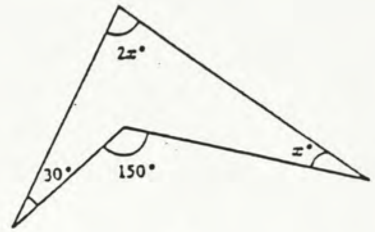


(37) In the figure above, $AGFE$ is a straight line, $AB \parallel CD$ and $BC \parallel DE$. Express c in terms of a and b.

A. $c = a + b$
B. $c = a + b - 90$
C. $c = a + b - 180$
D. $c = 180 - a + b$
E. $c = 180 - a - b$

(37) 在上圖中，$AGFE$ 爲一直線，$AB \parallel CD$ 及 $BC \parallel DE$。以 a 及 b 表 c。

A. $c = a + b$
B. $c = a + b - 90$
C. $c = a + b - 180$
D. $c = 180 - a + b$
E. $c = 180 - a - b$



(38) In the figure above, x =

A. 30
B. 35
C. 40
D. 45
E. 50

(38) 在上圖中，x =

A. 30
B. 35
C. 40
D. 45
E. 50

(39) If A is an acute angle, $\sin A = x$ and $\cos A = y$, which of the following statements is/are true ?

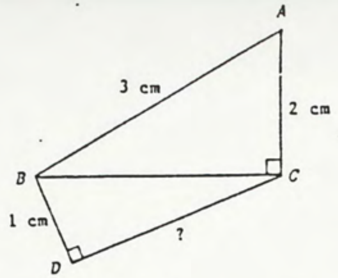
(I) $\frac{x}{y} = \tan A$
(II) $x + y = 1$
(III) $x^2 + y^2 = 1$

A. (I) only
B. (II) only
C. (III) only
D. (I) and (III) only
E. (II) and (III) only

(39) 若 A 是一銳角，$\sin A = x$ 及 $\cos A = y$，下列各項，何者爲真？

(I) $\frac{x}{y} = \tan A$
(II) $x + y = 1$
(III) $x^2 + y^2 = 1$

A. 只有(I)
B. 只有(II)
C. 只有(III)
D. 只有(I)及(III)
E. 只有(II)及(III)



(40) In the figure above, $\angle ACB$ and $\angle BDC$ are right angles. $CD =$

A. 2 cm
B. 4 cm
C. $\sqrt{5}$ cm
D. $\sqrt{12}$ cm
E. $\sqrt{14}$ cm

(40) 在上圖中，$\angle ACB$ 及 $\angle BDC$ 都是直角。$CD =$

A. 2 cm
B. 4 cm
C. $\sqrt{5}$ cm
D. $\sqrt{12}$ cm
E. $\sqrt{14}$ cm

– END OF PAPER –

– 測驗卷完 –

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KEY

1. B 2. C 3. A 4. C 5. C 6. E 7. C 8. B 9. B 10. A

11. C 12. C 13. B 14. B 15. D 16. D 17. D 18. E 19. E 20. A

21. E 22. D 23. E 24. D 25. A 26. E 27. E 28. B 29. D 30. D

31. A 32. C 33. A 34. C 35. E 36. C 37. E 38. C 39. D 40. A

Appendix B

Mathematical Reasoning Test

數學推理測驗

時間：40分鐘

問題解答說明

例題(a)

3 8 15 24 X 48
X = ?

Ans. (答案)

A. 30
B. 33
C. 35
D. 36
E. None of the above

Ans. C

這是一系數列。該數列中每項爲其前項加上一數。

各項所加的數亦自成系列，該數列各項所加的數依次爲3，5，7，9，11。數列中X爲第五系次，因此

$$X = 24 + 11 = 35$$

例題(b)

$(2X + 2)^2 = 4X^2 + 8X + 4$
$(3X + 2)^2 = 9X^2 + 12X + 4$
$(aX + b)^2 = a^2X + \triangle X + b^2$
$\triangle = ?$

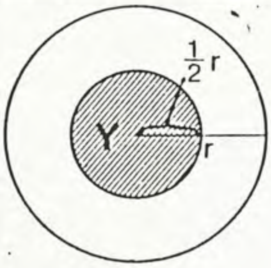
A. $2a$
B. $3b^2$
C. $a^2 + b^2$
D. ab
E. $2ab$

Ans. E

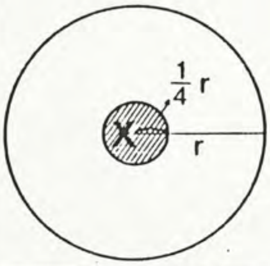
三者皆爲等式。每一等式的右邊中項係數是兩倍於左邊X項係數與常數之乘積。因此

$$\triangle = 2 \times a \times b = 2ab$$

例題(c)



The shaded (有線) area Y = $\frac{1}{4}$ of the whole circle.



The shaded area X = Z of the whole circle.
Z = ?

A. $\frac{1}{8}$
B. $\frac{1}{10}$
C. $\frac{1}{12}$
D. $\frac{1}{16}$
E. $\frac{1}{32}$

Ans. D

圓形面積 $= \pi \times (\text{半徑})^2$，故圓形面積與半徑平方成正比(如上圖)。
若下圖中斜綫圓形之半徑爲大圓形半徑之$\frac{1}{4}$，其面積X則爲大圓形面積之：

$$\left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

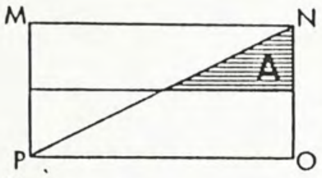
DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO DO SO.
聽到主試者吩咐，才可翻頁。

Students should read all the questions carefully. Illustrations (例示) and clues (提示) are given in the questions to show how the problems are tackled.

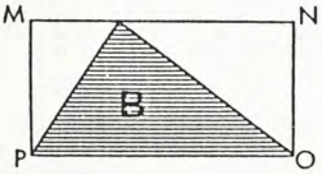
学生應細心閱讀各題，題目內均有例示及提示，作為解題的方式。

Ans. (答案)

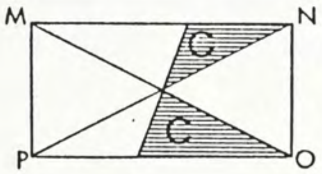
1.



The area of A = $\frac{1}{8}$ of the □ MNOP



The area of B = $\frac{1}{2}$ of the □ MNOP



The area of C = X of the □ MNOP

X = ?

A. $\frac{1}{3}$
B. $\frac{1}{4}$
C. $\frac{1}{5}$
D. $\frac{2}{5}$
E. None of the above

2. 37 11 48 22 59 33 X

X = ?

A. 70
B. 69
C. 60
D. 44
E. None of the above

3.

2 | 8	—	-2 | 5	=	-4 | 3
3 | -2	—	2 | 8	=	6 | -10
4 | 4	—	-4 | -2	=	?

A. 0 | 6
B. 8 | 2
C. -8 | 2
D. -16 | 6
E. -16 | -8

4.

$$1 + 3 = 2^2 = 4$$
$$1 + 3 + 5 + 7 + 9 + 11 = 6^2 = 36$$
$$1 + 3 + 5 + 7 + \ldots\ldots + 21 = X$$

X = ?

A. 169
B. 144
C. 121
D. 100
E. None of the above

Ans. (答案)

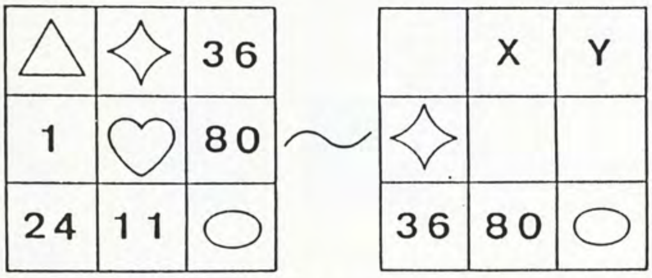
5. (5) (7) ( ) (11) (13)

3 15 35 X 99 143

X = ?

A. 50
B. 63
C. 65
D. 70
E. 81

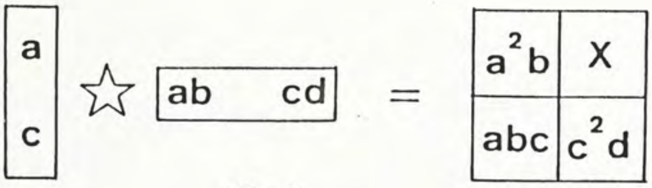
6.



Find X and Y

	X	Y
A.	△	1
B.	♡	11
C.	11	24
D.	1	24
E.	♡	△

7.



X = ?

A. a^2d
B. abc
C. ab^2
D. acd
E. cd^2

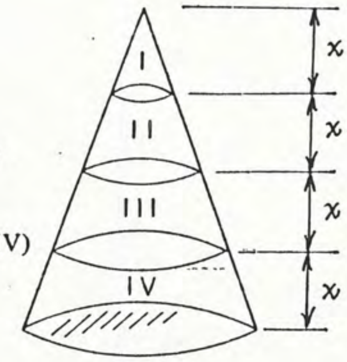
8. The cone (錐體) at the right shows

Volume (I) : Volume (I + II) = 1 : 8
(體積)

Volume (I) : Volume (I + II + III) = 1 : 27
(體積)

Then, Volume (I) : Volume (I + II + III + IV)

= ?



A. 1 : 36
B. 1 : 49
C. 1 : 64
D. 1 : 81
E. 1 : 125

9. If X < 6
and Y < 12
then, which of the following is true (正確)?

A. X > Y
B. X < Y
C. Y − X = 6
D. X = 2Y
E. X + Y < 18

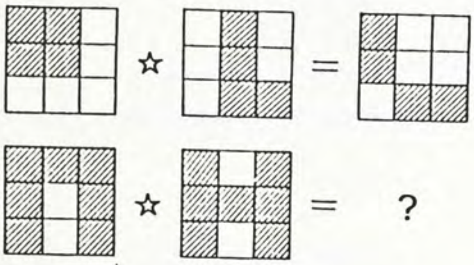
Ans. (答案)

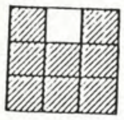
10.

$$2+4=2\times 3$$
$$2+4+6=3\times 4$$
$$2+4+6+8=4\times 5$$
$$2+4+6+8+\cdots\cdots+30=?$$

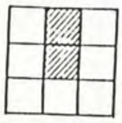
A. 5×6
B. 9×10
C. 15×16
D. 30×31
E. None of the above

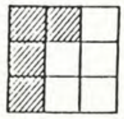
11.

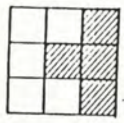


A.


D.


B.


E.


C.


12. If X is an integer (整數),
$2 < X < 7$
$5 < X < 10$
$4 < X < 8$
then X may be

A. exactly determined (一個解答)
B. any of 2 values (二個解答)
C. any of 3 values (三個解答)
D. any of 4 values (四個解答)
E. difficult to determine (不能解答)

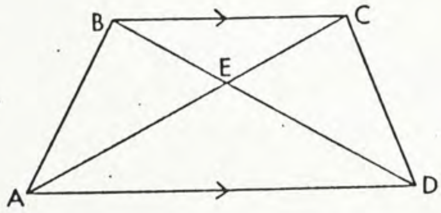
13. $-X\left(-X\left\{-X\left[-X(-X+2)+3\right]+4\right\}+5\right)+6$

$= -X^5+2X^4-3X^3+4X^2+\square+\triangle$

$\square = ?$
$\triangle = ?$

A. $\square = -5X^2$, $\triangle = 6X$
B. $\square = 5X^2$, $\triangle = 6X$
C. $\square = -5X$, $\triangle = -6$
D. $\square = 5X$, $\triangle = -6$
E. $\square = -5X$, $\triangle = 6$

14.

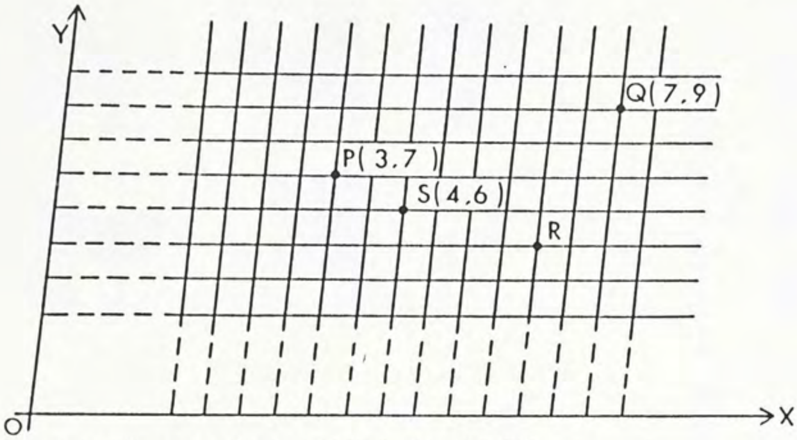


In the figure,
if $BC \parallel AD$
the area of $\triangle$ CAD = a
the area of $\triangle$ CED = b
the area of $\triangle$ CEB = c
then the area of $\triangle$ BEA = ?

A. $a-c$
B. $a-b$
C. b
D. $\frac{a+b}{2}$
E. $\frac{b+c}{2}$

Ans. (答案)

15.



What are the co-ordinates (座標) of point R?

A. (5, 3)
B. (4, 10)
C. (6, 5)
D. (6, 6)
E. None of the above

16.

If

4			X	
3				
2	X			
1				X
0				
−1		X		
−2				
−3				

$= 2 - y + 4y^2 + y^3$

then

4		X		
3				
2				
1	X			
0				X
−1				
−2				
−3			X	

= ?

A. $1 + 4y + 3y^2 + y^3$
B. $1 + 4y + 3y^3$
C. $1 + 4y + 3y^2$
D. $1 + 4y - 3y^2 - y^3$
E. $1 + 4y - 3y^2$

17.

$2^m + 3^n = 7$

$(-3)^m + 4^n = 13$

$(-2)^m + 5^n = ?$

A. 3
B. 9
C. 17
D. 19
E. 21

18.

If

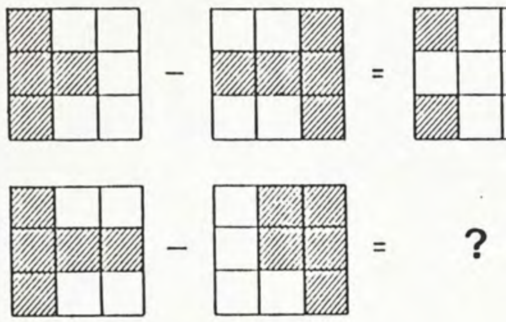
8	7	= 56	X = 4
X	5	= 20	Y = 2
4	Y	= Z	Z = 8

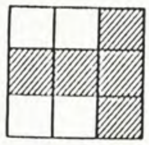
then

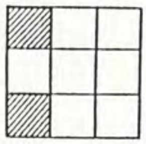
5	−24	= −120	
X	1	= 4	Z = ?
1	Y	= Z	

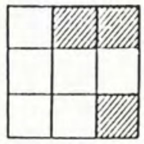
A. 23
B. − 22
C. −24
D. −25
E. 25

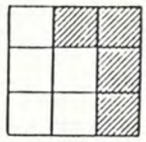
Ans. (答案)

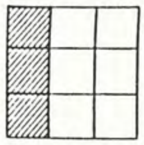
19. 

A. 

B. 

C. 

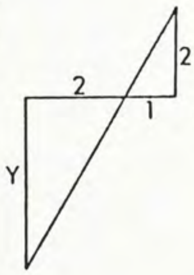
D. 

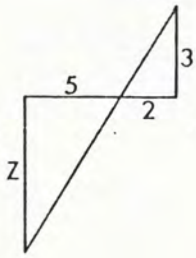
E. 

20. If ◇ ◇ ◇ = □ □
and □ □ □ □ = ▭ ▭

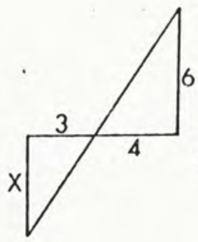
then ◇ ◇ ◇ ◇ = How many ▭ ?
(多少個这種圖形)

A. $\frac{3}{4}$
B. 1
C. $1\frac{1}{2}$
D. $1\frac{1}{4}$
E. None of the above

21. If 

Y = 4, and 

Z = 7.5,

then 

X = ?

A. 2
B. 2.5
C. 4
D. 4.5
E. 5

22. If in 2 days 2 men make 2 tables, then in 4 days 4 men make X tables.

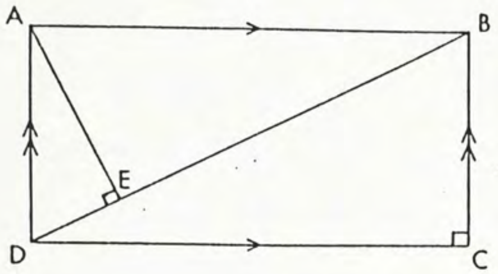
X = ?

(假如 2人2日內可製造 2張枱，
則 4人4日內可製造X張枱。 X=？)

A. 4
B. 6
C. 8
D. 16
E. None of the above

Ans. (答案)

23.



AE x BD = ?

A. BC x CD
B. BC x BD
C. BD x CD
D. BD x AD
E. BE x AB

24.

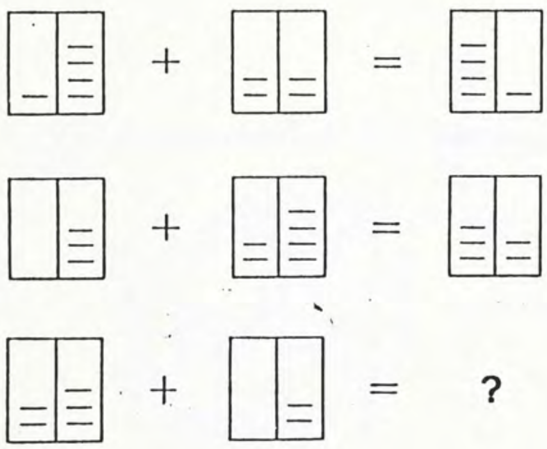
$$1^3 + 2^3 = 3^2$$

$$1^3 + 2^3 + 3^3 = 6^2$$

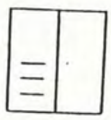
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = ?$$

A. 12^2
B. 15^2
C. 18^2
D. 21^2
E. None of the above

25.

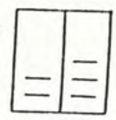


A.



B.

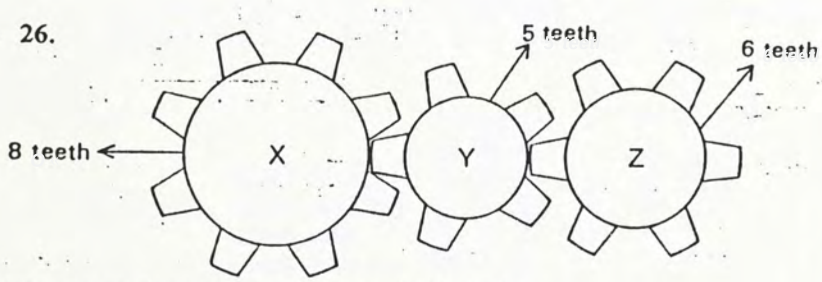
D.



C.

E. None of the above

26.



The above figure shows 3 gears (齒輪), each with different number of teeth.
When X has made a complete turn (全轉一圈), how many turns (圈) will Z make?

(上圖顯示3個齒輪，每個齒數均不同。
若X全轉一圈，Z轉了多少個圈？)

A. $\frac{3}{4}$
B. $\frac{4}{3}$
C. 1
D. $1\frac{1}{2}$
E. 2

Ans. (答案)

27. $\frac{1}{x^2} + 1 = y$

If $y > 100$

$x = ?$

A. $x > 1$
B. $x = 1$
C. $1 > x > 0$
D. $-1 > x > -5$
E. None of the above

28.

2	6	6	2			
2	8	12	8	2		
2	10	20	20	10	2	
2	12	X	Y	X	12	2

Find X and Y

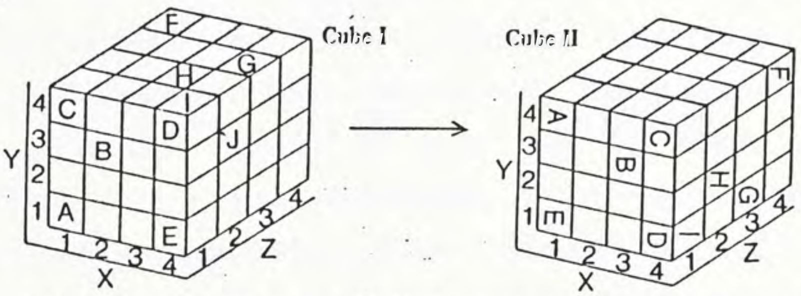
A. X = 30, Y = 40
B. X = 32, Y = 40
C. X = 32, Y = 30
D. X = 22, Y = 30
E. X = 14, Y = 22

29.
$$\begin{array}{r} 4\,3\,1\,4 \\ +\ 2\,5\,2\,1 \\ \hline 1\,0\,1\,3\,5 \\ -\ \ 4\,X\,5\,3 \\ \hline 2\,4\,5\,2 \end{array}$$

Then X = ?

A. 2
B. 3
C. 4
D. 5
E. 6

30. Cube I is rotated (轉動) 90° to the position of Cube II as shown in the figure below:



In the coordinate (座標) system (X, Y, Z), small cube C in Cube I located (位置在) in position (1, 4, 1) is in the 1st block of X coordinate, in the 4th block of Y coordinate, and in the 1st block of Z coordinate. Thus,

	In Cube I (X, Y, Z)		In Cube II (X, Y, Z)
Small cube B is	(2, 3, 1)	→	(3, 3, 1)
Small cube G is	(4, 4, 3)	→	(4, 1, 3)
Small cube F is	(1, 4, 4)	→	(4, 4, 4)
Small cube J is	(4, 3, 2)	→	?

A. (4, 1, 2)
B. (2, 2, 2)
C. (1, 3, 2)
D. (3, 1, 2)
E. (2, 1, 3)

STOP. YOU MAY CHECK YOUR WORK ON THIS TEST.

KEY

1. B	2. A	3. D	4. C	5. B	6. D	7. D	8. C	9. E	10. C
11. B	12. A	13. E	14. C	15. C	16. E	17. B	18. D	19. E	20. E
21. D	22. C	23. A	24. D	25. B	26. B	27. C	28. A	29. B	30. D

Appendix C

Instructional Material

The pamphlet issued to the students for instruction is included here for reference. It consists of four units, each to be covered in a one-hour session:

Unit 1 Sequence and Arithmetic Progression
Unit 2 Terms in an Arithmetic Progression
Unit 3 Summation of an Arithmetic Progression
Unit 4 More Examples on Arithmetic Progressions

In actual implementation, Exercise 1.2 of Unit 1 (p.9) was deferred until the second session for both groups owing to problem of time allocation. The exercise was done as a revision for material covered in Unit 1.

In Unit 1, the notions of 'sequence' and 'arithmetic progression' were introduced with examples. The different patterns found in sequences were highlighted. The two essential characteristics of an arithmetic progression (A.P.), namely the first term and the common difference, were stressed. They were identified before the general term of an A.P. can be found. The focus was on a concrete representation of an A.P. showing the differences between terms. The formula for finding the general term of an A.P. was then derived firstly from the pattern observed and alternatively from the concrete representation.

In Unit 2, problems using only the general term formula were discussed. Solution were worked out clearly to show the structure of the problems. The concrete representation was referred to whenever it led to a better understanding of the problem. The main theme of this unit was problems involving only the terms and their order.

Unit 3 was motivated by the story of Carl Friedrich Gauss. The pattern that the sums of particular pairs of terms remained constant was emphasized before deducing the formulas for summation. This would guarantee a better understanding of the complicated formula evolved. In fact, a figure using flights of steps to represent A.P.'s illustrated the pattern to impress the students. Then, problems involving direct application of these summation formulas were discussed.

More examples on arithmetic progressions were discussed in Unit 4. Contents and formulas covered in the previous three units would be involved in this session. The focus was on the analysis of problems and the application of results on the arithmetic progressions. This unit also summarized the material covered, providing an overall review of the topic.

Structural diagrams were used for the treatment group only. These diagrams were shown in blocks bounded by dotted lines in the subsequent pages. This was the only difference in the instructional material between the two groups. Structural diagrams for some of the examples were not shown and were to be drawn by the students in class.

Solutions to worked examples were also not shown. These were either presented by the instructor in class or to be attempted by the students themselves before class discussion.

School: ______________________

Class : Form 3 (   )

Name : ____________________ (   )

* * * * * * * * * * *

ARITHMETIC PROGRESSION (A.P.)

Notes to the Students:

This course on arithmetic progressions will consist of 4 units, each to be covered by a one-hour lesson. The text material for the lessons provides the definitions of terms, formulae, and problems for demonstration. Students are expected to fill in the details during the lessons. There are also problems which are to be done as classwork. Brief solutions to these exercises will be given at the end of each lesson.

UNIT ONE SEQUENCE AND ARITHMETIC PROGRESSION

1.1 Sequence

Observe the following lists of numbers. You may discover some patterns. Try to guess the next number.

(a) 1, 3, 5, 7, ... ________

(b) 3, 6, 9, 12, 15, 18, ... ________

(c) 8, 88, 888, 8888, 88888, ... ________

(d) 1, 4, 9, 16, 25, ... ________

(e) 5, 8, 11, 14, 17, ... ________

Each of these lists of numbers is called a sequence <數列>. Each number in the sequence is called a term <項>. For example, in (a) above, 1 is the first term; 3 is the second term; 5 is the third (3rd) term; etc.

Now, for the sequence 1, 4, 9, 16, 25, ..., what would you guess for the 8th term and the 13th term ?

Ans. the 8th term = ________

the 13th term = ________

Usually, we write these terms in symbols as

T_8 and T_{13} respectively.

Thus, $T_1 = 1$, $T_2 = 4$, $T_{10} =$ ________.

Some sequences may have patterns that are easy to discover. But you can never know for sure unless you actually get the rule of computing the terms in a particular sequence. Now, for instance, I get a rule for generating numbers and I keep it as a secret. The sequence is as follows: 1, 2, 3, 4,..... Guess T_5, i.e. the next number in the sequence.

Your guess is ________.

My answer is that the 5th term is ________.

Are you right ?

This would NOT be so if the rule of calculating terms is explicitly given. For example, a sequence of numbers is such that its nth term can be computed by the formula $3n + 1$. Complete the following:

T_1 = ________

T_2 = ________

T_3 = ________

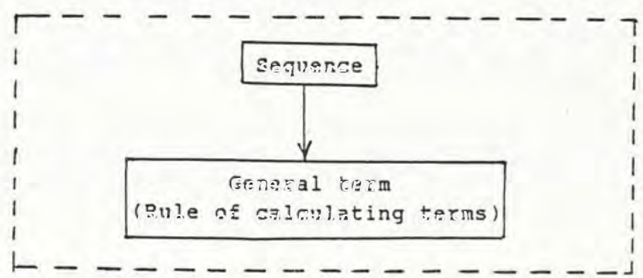
T_{24} = ________

T_{101} = ________

Can you be sure that you are correct ?

In this case, we say that the general term <通項> is $T_n = 3n + 1$.

That is, the general term gives a rule for finding each term of a sequence. If this formula is given, any term can be computed. If conversely, only some terms are given, we can never be sure about the general term without further information. We shall in this course study a simple kind of sequences known as arithmetic progressions <算術級數>.



Example 1.1.1

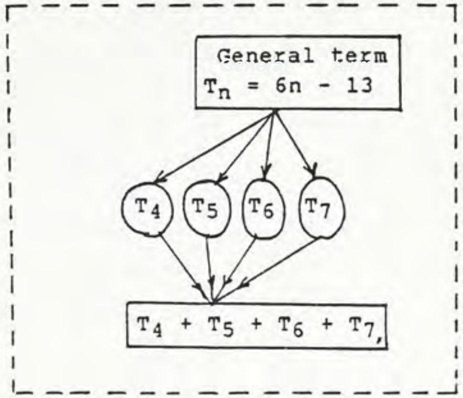
The general term of a sequence is given by

$$T_n = 6n - 13 .$$

Write down the 4th to 7th terms.

Hence find the following sum: $T_4 + T_5 + T_6 + T_7$.

Solution



*Remark: If you are asked to find

$$T_1 + T_2 + \ldots + T_{100}$$

or $T_8 + T_9 + \ldots + T_{67}$,

can you find them ?

We shall learn some formulae to do these when these terms are from arithmetic progressions.

Exercise 1.1

Write down the first five terms of each of the following sequences with

(a) $T_n = 4n - 7$ ____, ____, ____, ____, ____

(b) $T_n = (-1)^n$ ____, ____, ____, ____, ____

(c) $T_n = n^2 - n$ ____, ____, ____, ____, ____

1.2 Arithmetic Progression (A.P.)

From this section onwards, we shall study a particular type of sequences having some very simple patterns. Consider the following sequences of numbers:

(a) -1, 5, 11, 17, 23, ...

(b) 3, 6, 9, 12, 15, ...

(c) 20, 13, 6, -1, -8, ...

(d) 1, 3, 5, 7, 9, ...

What do you notice for each of these sequences ?
Do they have some common property ?
Question: Do you think the sequence 1, 4, 9, 16, 25, ... also has this property ? Ans. Yes/No

In each sequence above, we observe that any term differs from its preceding one by a constant. For example, in (a):

$$5 - (-1) = 6$$
$$11 - 5 = 6$$
$$17 - 11 = 6$$
$$23 - 17 = 6$$

That is, in these sequences, we have

$$T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \ldots .$$

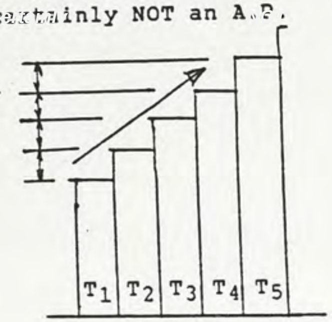
This type of sequences is called an arithmetic progression < 算術級數 >, which is usually abbreviated to A.P.

That is to say, an arithmetic progression is a sequence in which every term (except the first) is obtained by adding a fixed constant to the preceding term. This fixed constant is called the common difference < 公差 > of the A.P.

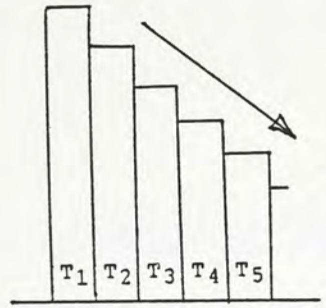
When the common difference is 0, we have trivial A.P.'s. For example, 5, 5, 5, 5, 5, ... is an A.P.

The common difference can be positive or negative.
In (a), the common difference is 6. The A.P. increases.
In (c), the common difference is ____. The A.P. ________.

So, in general, an A.P. either increases uniformly or decreases uniformly. The sequence 7, 12, ..., 3, ... is certainly NOT an A.P.



common difference +ve



common difference -ve

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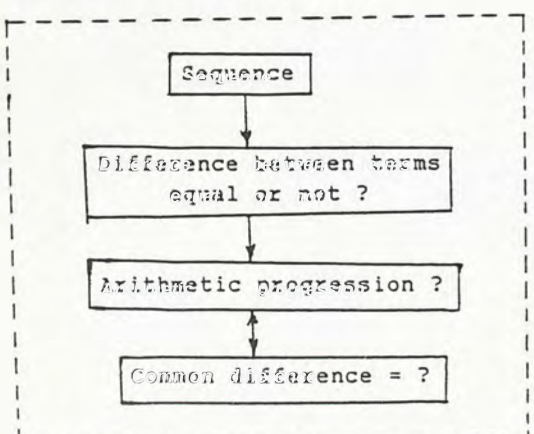
Example 1.2.1

Check whether the following sequences are A.P.'s or not. If yes, state also the common difference:

(a) 1, 4, 7, 11, ...
(b) 70, 68, 66, 64, ...
(c) 5, 9, 13, 17, ...
(d) 8, 1, -6, -13, ...
(e) x + 15, x + 19, x + 23, x + 27, ...
(f) y, y - 5, y + 7, y + 9, ...

Here, x and y are some unknown constants.

Solution



(a)

(b)

(c)

(d)

(e)

(f)

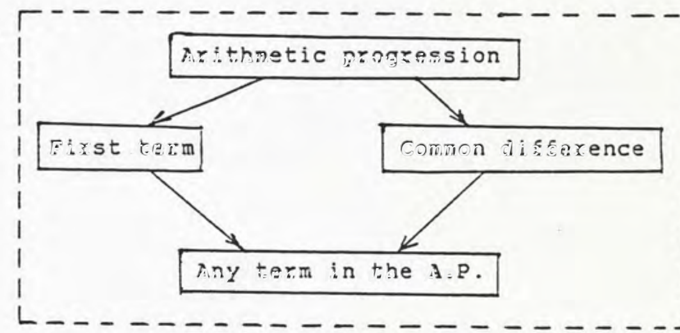
Example 1.2.2

The following sequence is known to be an A.P.:

-1, 5, 11, 17, ...

(a) Find its 10th term.
(b) Find its general term, i.e. a formula to compute T_n .

Solution



(a)

$T_1 = -1$
$T_2 = 5 = (-1) +$
$T_3 = 11 = (-1) + \quad +$
$T_4 = 17 = (-1) +$
....
$T_{10} =$

I----I----I----I----I ..... ---I----I----I
T_1 T_2 T_3 T_4 T_{10}

(b)

*Remark: We see that once we know the first term and the common difference of an A.P., we can find any term of that A.P.

In general, we can find a formula for the general term of an A.P. as follows.

Let $a = T_1$,and $d =$ common difference .

Then $T_1 = a$

$T_2 = a + d$

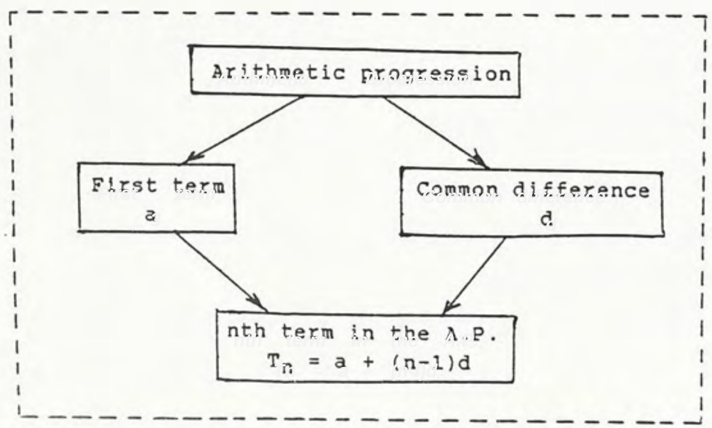
$T_3 = (a + d) + d = a + 2d$

$T_4 = (a + 2d) + d = a + 3d$

....

Therefore, $\boxed{T_n = a + (n-1)d}$

```
I----I----I----------------------------------------I----I
T1   T2   T3                                             Tn
  d    d     d    .....                             d
```

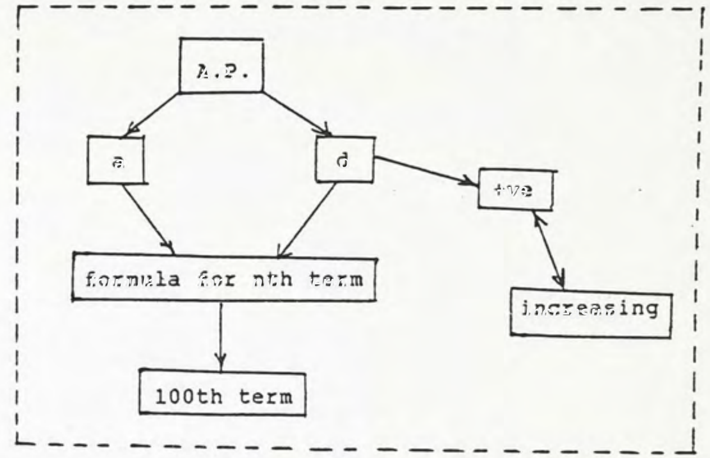


Example 1.3.3

The numbers -11, -8, -5, -2, ... Find the 100th term.

*Remark: A student said that the answer is -79. Do you think this answer is possible ?

Solution



In this given A.P., first term = , i.e. a,

and common difference = , i.e. d.

Exercise 1.2

1. Check whether each of the following sequences is an A.P. or not. If yes, write down also the common difference.
   (a) 10, 5, -2, -7, ...
   (b) 4, 8, 12, 16, ...
   (c) 18.5, 17, 15.5, 14, ...
   (d) -2, 2, 6, 10, ...
   (e) $x - 8$, $x - 3$, $x + 2$, $x + 7$, ...
   (f) k, $3k$, $5k$, $7k$, ...

   Here x and k are some unknown constants.

2. The first term and the second term of an A.P. are 17 and 29 respectively.
   (a) Find the 9th term and the 21st term.
   (b) Write down the general term T_n in terms of n.

3. The common difference of an A.P. is 3. The 10th term is 37. Find the 1st term and the 5th term.

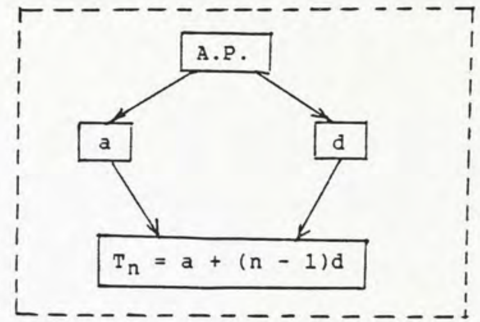
4. In the A.P. 39, 33, 27, 21, ..., it is known that the kth term is -93. What is k ?

5. Suppose -6, x, y, 9 form an A.P. Find x and y.

UNIT TWO TERMS IN AN ARITHMETIC PROGRESSION

In this unit, we shall study some examples which involve the terms in arithmetic progressions. We shall see how these problems can be solved with the formula we learned in Unit One.

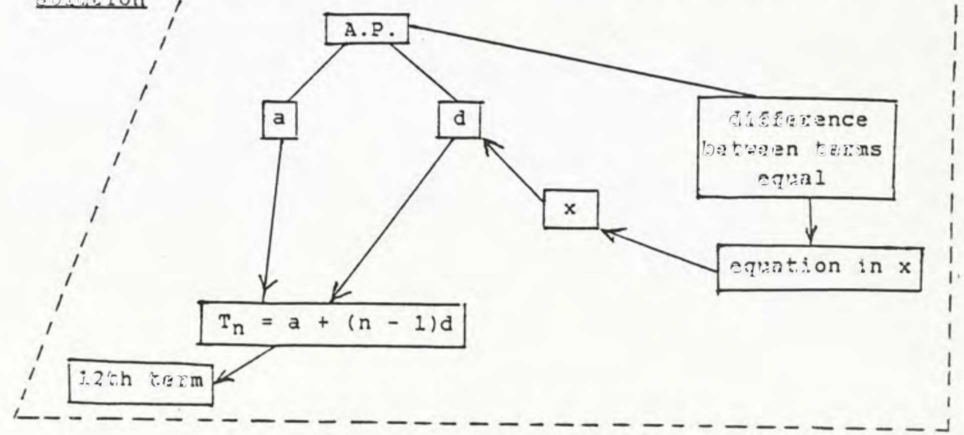
For an A.P. with a = 1st term and d = common difference, $T_n = a + (n - 1)d$



Example 2.1

The numbers -9, x, $(1 - 2x)$, ... form an A.P.
(a) Find the value of x.
(b) Find the nth term and the 12th term.

Solution

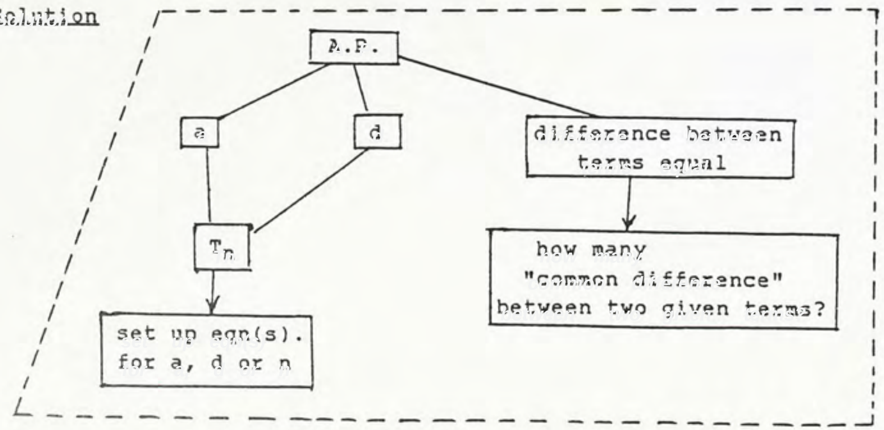


*Remark: When 3 or more terms in an A.P. are given, they always bear a simple relationship between one another, i.e. $T_2 - T_1 = T_3 - T_2 = \dots$.

Example 2.2

Suppose -12, p, q, r, s, 8 is an A.P.
Find p, q, r and s.

Solution



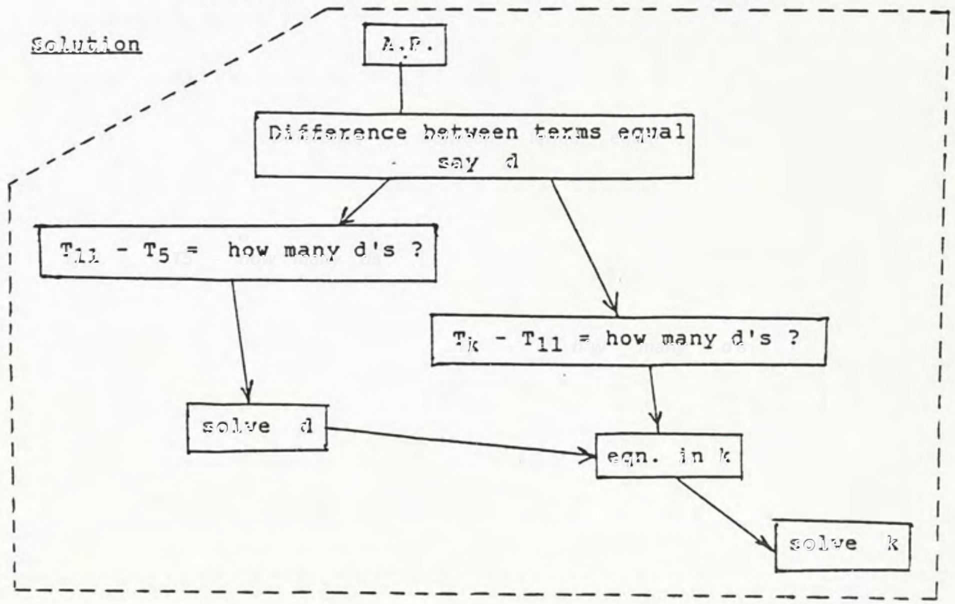
*Remark: We can have two different ways of looking at the problem.

Example 2.3

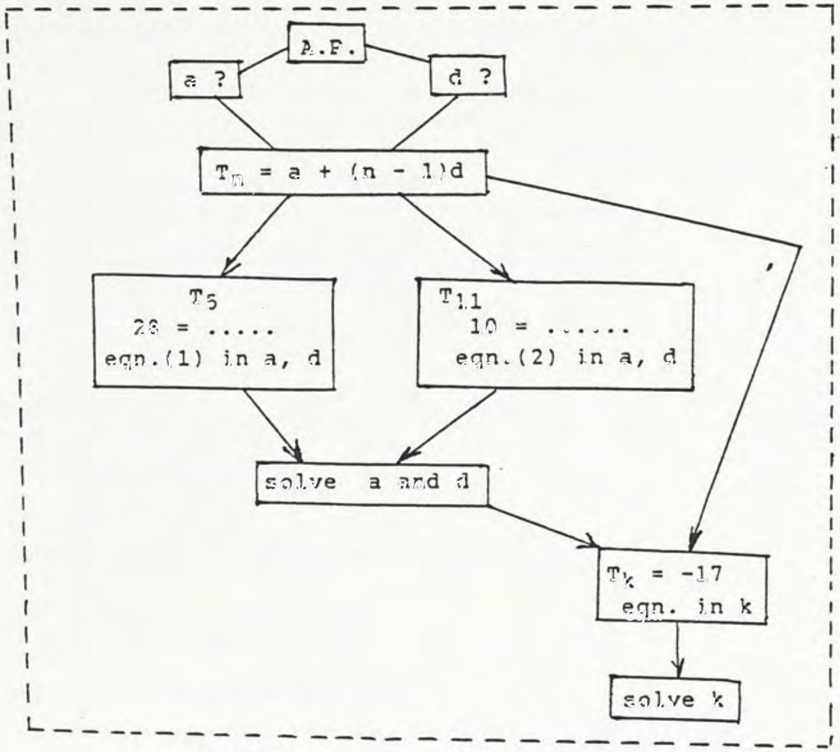
The 5th term and 11th term of an A.P. are 28 and 10 respectively.
(a) Find the 23rd term.
(b) If the kth term is -17, what is the value of k ?

Question: (i) Can the answer to (a) be 32 or 14 ?
(ii) Can k be 7 ?

Solution



Alternatively, we have



*Remark: (a) Again, you can have two different ways of finding a and d. So, problems may not necessarily be solved in only one way.

(b) Notice that k denotes the order of the term, not the value of the term.

Exercise 2

1. If 6, x +2, x - 1 are in A.P., find x and the 10th term.

2. The sequence 5, x, y, z, 29 is an A.P.
   Find the value of x, y and z.

3. In an A.P., the 2nd term is 4 and the 9th term is 39.
   Find the 21st term.

4. Given an A.P. 10, 13, 16, ..., 94.
   How many terms are there totally ?

5. If x_1, x_2, x_3, x_4, ... is an A.P.,
   do you think the sequence

   x_2, x_5, x_8, x_{11}, ...

   is also an A.P. ? Why or why not ?

UNIT THREE SUMMATION OF AN ARITHMETIC PROGRESSION

In this unit, we shall derive a formula for the summation of the terms of an A.P.

** Story of Carl Friedrich Gauss **

Let us first consider a simple example.
Suppose we are asked to find the sum of the first ten terms in the A.P. 1, 2, 3, 4, ...
Thus let S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10
Reversing the order of the sum, we have

S = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1

Adding these two expressions, we have

$$\begin{array}{rrl} & S = & 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ +) & S = & 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 \\ \hline & 2S = & 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 \\ & 2S = & 10 \times 11 \\ & S = & 10 \times 11 / 2 = 55 \end{array}$$

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Now, consider an A.P.

Let a = 1st term
d = common difference
S_n = sum to n terms of the A.P.
[Note: e.g. $S_4 = T_1 + T_2 + T_3 + T_4$]

We can derive a formula for S_n in a similar way:

$$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$
$$= a + (a+d) + (a+2d) + \ldots + [a+(n-2)d] + [a+(n-1)d]$$

Reversing the order of the terms and adding, we have:

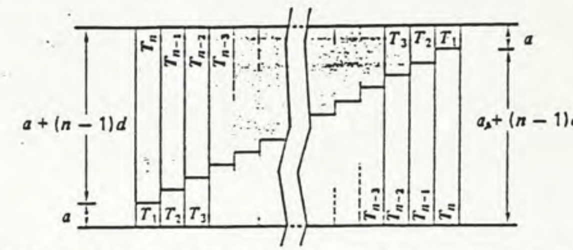
$$\begin{array}{rrl} & S_n = & a + (a+d) + \ldots + [a+(n-2)d] + [a+(n-1)d] \\ +) & S_n = & [a+(n-1)d] + [a+(n-2)d] + \ldots + (a+d) + a \\ \hline & 2S_n = & [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] + [2a+(n-1)d] \\ & = & n \times [2a+(n-1)d] \end{array}$$

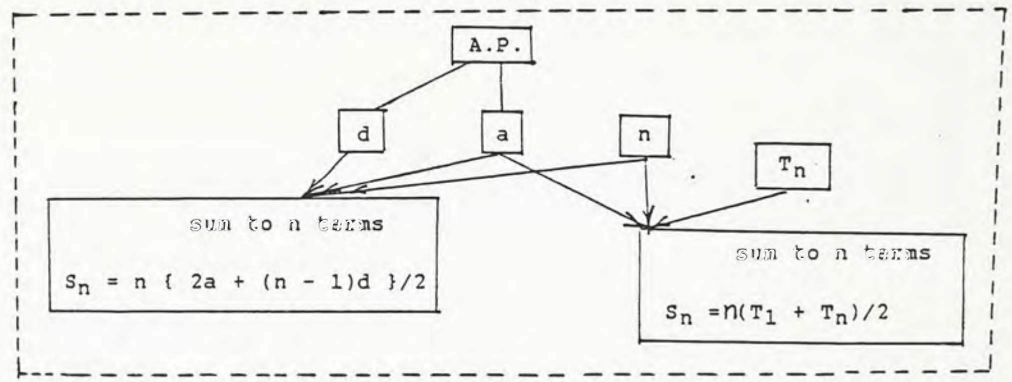
$\therefore \quad \boxed{S_n = \frac{1}{2} n [2a + (n-1)d]}$

Now, $T_1 + T_n = a + [a + (n-1)d] = 2a + (n-1)d$.

Therefore, we also have $\boxed{S_n = \frac{1}{2} n (T_1 + T_n)}$

The above procedure of addition is illustrated below:

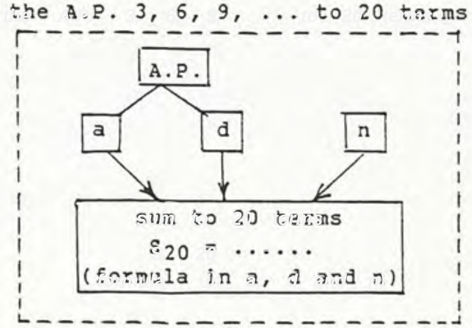




Example 3.1

Find the sum of the A.P. 3, 6, 9, ... to 20 terms.

Solution

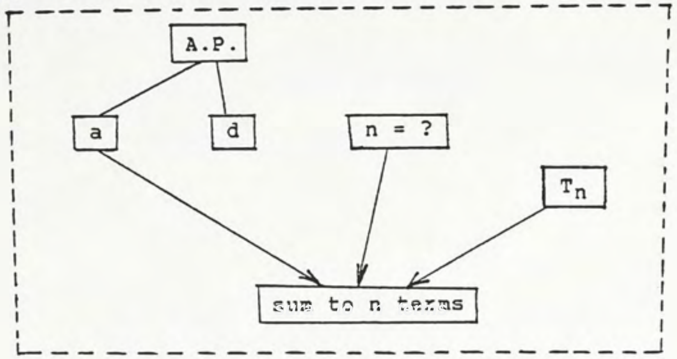


a =
d =
n =
S_{20} =

Example 3.2

Find the sum of 1 + 3 + 5 + 7 + ... + 201.

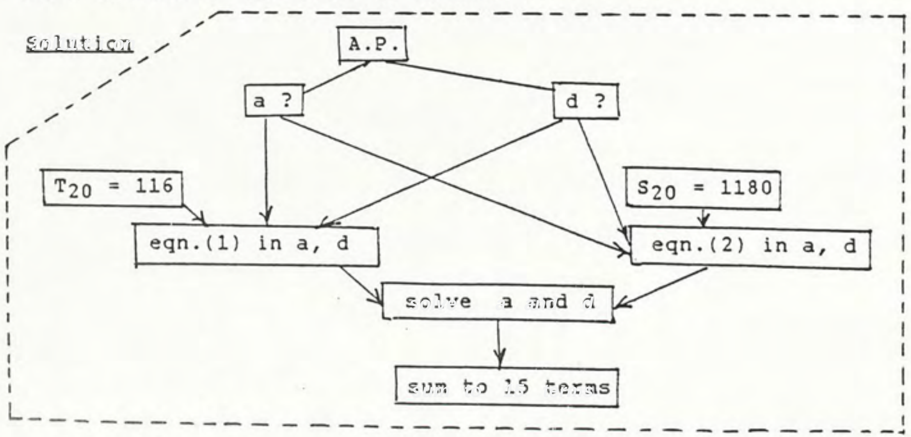
Solution



Example 3.3

The 20th term of an A.P. is 116.
The sum of the first 20 terms is 1180.
Find the sum of the first 15 terms.

Solution



*Remark: In this kind of problems, notice that a and d are important intermediate unknowns.

Example 3.4

Find the sum of all the multiples of 9 that are less than 200.
〈倍數〉

Solution

Exercise 3

1. Find the sum to 8 terms of the A.P.
   $-11, -8, -5, \ldots$.

2. Find the sum of the first 25 terms of the A.P.
   $20, 16, 12, \ldots$
   Why do you think the answer is negative ?

3. Find the sum of the A.P.
   $42, 39, 36, \ldots, -36$.

4. The first term of an A.P. is -5.
   The sum to 17 terms is 85.
   Which term has the value 0 ?

5. Find the sum of all the multiples of 3 between 1 and 110.

6. Given the A.P. $8, 11, 14, \ldots$
   Find the sum of the terms from the 10th to the 20th.

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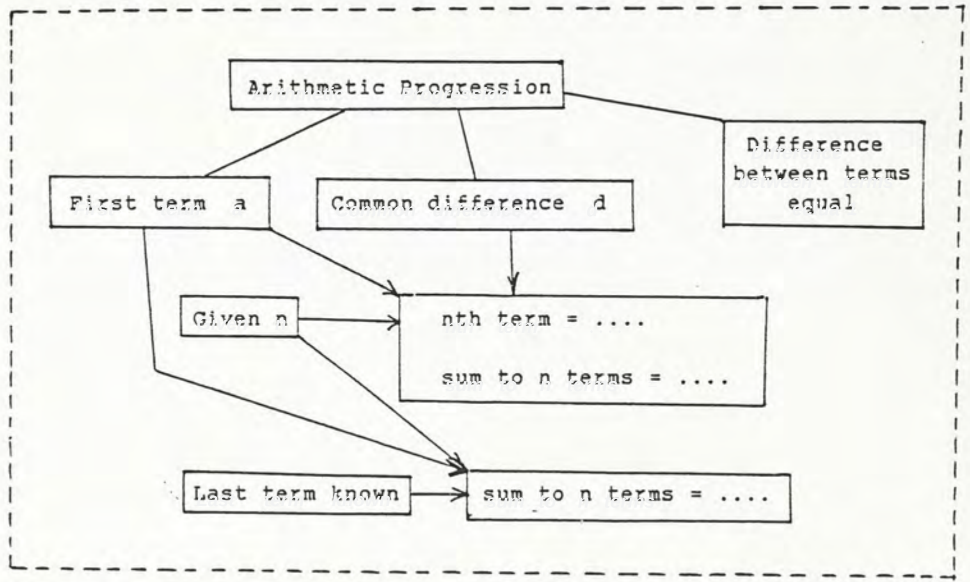
UNIT FOUR MORE EXAMPLES ON ARITHMETIC PROGRESSIONS

In the previous two units, we have derived the following formulae for A.P.'s:

A.P. with a = 1st term
d = common difference
$$T_n = a + (n - 1)d$$
$$S_n = n\,[\,2a + (n-1)d\,]/2$$
$$S_n = n\,(T_1 + T_n)/2$$



Example 4.1

The sum of the first 4 terms of an A.P. is 20.
The sum of the next 4 terms is 60.
Find the sum of the first 20 terms.

Solution

Example 4.2

The sum of the first 3 terms of an A.P. is -9.
If the 5th term is greater than the first term by 8, find the sum of the first 15 terms.

Solution

Example 4.3

Paul works in a factory at a starting salary of \$ 3500 per month. At the end of each year, he will receive an increment of \$ 100 per month in salary.

(a) Find the total income of Paul for the first 5 years.

(b) Another person, Tom, works in the same factory (starting at the same time as Paul) with a starting salary of \$3200 per month and an annual increment of \$ 150 per month.

(i) When will Tom's monthly salary be the same as Paul's ?

(ii) After how many years will the total income of Tom be equal to that of Paul ?

Solution

Exercise 4

1. Find the number of terms in the A.P.

$$1 + 3 + 5 + 7 + \ldots$$

that must be taken to make the sum equal to 100.

2. The A.P. 7, ..., 35 has a sum of 168.
Find the number of terms in this A.P. and its common difference.

3. The 5th term of an A.P. is 3.
The sum of the first 20 terms is -270.
Find the first term, the common difference and the sum to 40 terms.

4. The 5th term of an A.P. is 14 and the 18th term is -25.
Find the sum of the first 25 terms.

5. A supermarket is running a contest in which they will send the winner some money everyday for one month. The amounts are \$ 100 the first day, \$ 200 the second day, \$ 300 the third day, and so on. If the month has 30 days and you win the contest, how much money will the supermarket send you altogether ?

Appendix D

Problem-Solving Test (with Scoring Scheme)

The problem-solving test consists of two sections. The first section (Problems 1 to 8) and the second section (Problems 9 to 14) are on routine and nonroutine problems respectively. Problems in the routine section carry 5 marks each. Those of nonroutine type carry 8 marks each. A scoring scheme is given for solution to each problem. Alternative solutions are separated by dotted lines.

There are 3 types of marks allocated to the solution. 'A' means points allocated for expressions, equations or answers correct in every detail. 'M' means points allocated for the correct methods or formulas used although values substituted or some intermediate steps may not be all correct. For nonroutine problems (i.e. Problems 9 to 14), one point ('1P') is awarded for a solution that is well-planned or is working according to some relevant procedures even though the final goal may not be attained. This point will not be awarded if formulas and equations are written down without a reasonable objective or the final solution is actually obtained by some trials or by chance. Explanatory notes for these allocations in each problem are given in brackets.

TEST ON ARITHMETIC PROGRESSIONS

Time allowed: 1 hr. 15 min. There are 14 questions in this test. Answer ALL questions in the space provided. You may use calculators.	時限：1小時15分 本試卷有14題， 請在各題之空位上解答. 可用計算機.

1. Check whether each of the following given sequences is an A.P. or not. Write ' ' if yes and ' ' if no.

以下数列何者為算術級数？是的請填「✓」，不是的填「×」。

(a) 5, -10, 20, -40, ... (a) ×

(b) -17, -26, -35, -44, ... (b) ✓

(c) 42, 45, 48, 51, ... (c) ✓

(d) 1, 4, 9, 16, ... (d) ×

(e) x -1, x + 3, x + 7, ... (e) ✓ 1A each

2. The common difference of an A.P. is 4. The 10th term is 52. Find the first term of this A.P.

某算術級数的公差是4，而第10项為52。求此算術級数的第一项。

Solution

$T_n = a + (n - 1)d$ 1A (use of this formula)

$52 = a + (10 - 1)\cdot 4$ 1M + 1A (1M for sub. d = 4 and equating)

$52 = a + 36$ 1M (solving eqn.)

$\therefore a = 16$ 1A

$\underbrace{a,\ a+4,\ \ldots,\ a + 9\times 4}_{T_{10}}$ 1A (recurrent pattern)

$\therefore 52 = a + 9\times 4$ 1M (equating) + 1A

$52 = a + 36$ 1M (solving eqn.)

$\therefore a = 16$ 1A

3. How many terms are there in the following A.P.: -5, -2, 1, ... , 22 ?

以下之算術級数共有多少项：-5, -2, 1, ..., 22 ？

solution

Consider this A.P. as having n terms. 1M (eg. Let n be ..., use n in the eqn., etc.)

$a = -5,\ d = 3$ 1A

$22 = -5 + (n - 1)\cdot 3$ 1A (sub. to form eqn.)

$27 = 3(n - 1)$	1M (solving eqn.)
$9 = n - 1$	
$\therefore n = 10$	1A

Consider this A.P. as having n terms.	1M (similar to the above)
$a = -5$, $d = 3$	1A
$n = \frac{22 - (-5)}{3} + 1$	1M + 1A (sub. values in the first fractional expression)
$\therefore n = 10$	1A

Find the number of terms to be 10 by direct counting, eg. listing out part or all the terms.	2M + 1A (deduct 1A for minor mistakes)

4. Given an A.P. 8, 15, 22, ...
If its 21st term is $(3x + 76)$,
find the value of x.

給定一算術級數 8, 15, 22,...
若其第21項是 $(3x+76)$,
求 x 之值.

Solution

$a = 8$, $d = 15 - 8 = 7$	1A
$T_{21} = 8 + (21 - 1)\cdot 7$	1M (sub. into correct formula)
$= 8 + 140$	
$= 148$	1A
$148 = 3x + 76$	1M (equating to form eqn.)
$72 = 3x$	
$\therefore x = 24$	1A

5. The 4th term of an A.P. is 11.
Its 7th term is 35.
Find its 18th term.

某算術級數之第4項為11,
第7項為35.
求其第18項.

Solution

$T_7 = 35$

$T_4 = 11$

$\therefore 3d = 24$ 1M (subtracting)

$d = 8$ 1A

$11 = a + (4 - 1)\cdot 8$

$11 = a + 24$

$a = -13$ 1A

$T_{18} = -13 + (18 - 1)\cdot 8$ 1M (sub. the values a & d)

$= 123$ 1A

$T_7 = 35$

$T_4 = 11$

$\therefore 3d = 24$ 1M (subtracting)

$d = 8$ 1A

$T_{18} = T_7 + 11d$ 1M (use of this idea)

$= 35 + 11 \times 8$ 1A

$= 123$ 1A

$\left.\begin{array}{l} T_7 = 35 = a + 6d \\ T_4 = 11 = a + 3d \end{array}\right\}$ 1M (set up eqns.)

$\therefore 24 = 3d$

$d = 8$ 1A

$11 = a + 3 \times 8$

$a = -13$ 1A

$T_{18} = -13 + (18 - 1)\cdot 8$ 1M

$= 123$

6. Given an A.P. 35, 22, 9, ..., find the sum of the first 10 terms.

给定一算術級数 35, 22, 9, ... 求其前10项之和.

Solution

$a = 35,\ d = 22 - 35 = -13$	1A
$S_n = \frac{n}{2}[2a + (n-1)d]$	2M (correct formula)
$S_{10} = \frac{10}{2}[2\times35 + 9\times(-13)]$	1A
$= 5\times(-47)$	
$= -235$	1A

$a = 35,\ d = -13$	1A
$T_{10} = 35 + 9\times(-13)$	1M (to find T_{10})
$= -82$	
$S_n = \frac{n}{2}(T_1 + T_n)$	1M
$S_{10} = \frac{10}{2}(35 - 82)$	1A
$= -235$	1A

Obtain the correct sum by direct listing of terms to add.	2M + 1A (deduct 1A for minor mistakes)

7. The numbers 6, x, (33 - x), ... form an A.P. What is the value of x ? Find the difference between the 17th term and the 25th term.

一列数 6, x, (33-x), ... 為算術級数. x 是多少? 求其第17项與第25项之差.

Solution

$x - 6 = (33 - x) - x$	1A
$x - 6 = 33 - 2x$	
$3x = 39$	
$x = 13$	1A

$T_{25} - T_{17} = 8 \times \text{common difference}$	1M
$= 8 \times (13 - 6)$	1A (correct sub.)
$= 56$	1A

$x - 6 = (33 - x) - x$	1A
...	
$x = 13$	1A
$d = 13 - 6 = 7$	
$T_{25} = 6 + (25 - 1)\cdot 7 = 174$	1M + 1A
$T_{17} = 6 + (17 - 1)\cdot 7 = 118$	(1M for finding the terms)
$T_{25} - T_{17} = 174 - 118 = 56$	1A

8. The common difference of an A.P. is 3. The 9th term is twice its 4th term. Find the 26th term of this A.P.

某算術級數之公差為3，其第9項是第4項的兩倍。求此級數之第26項。

Solution

$d = 3$	
$T_4 = a + 3d = a + 9$	1M + 1A
$T_9 = a + 8d = a + 24$	(1M for expressing terms in terms of a and d)
$T_9 = 2\,T_4$	
$a + 24 = 2(a + 9)$	1M (equating)
$a + 24 = 2a + 18$	
$\therefore\ a = 6$	1A
$T_{26} = a + 25d$	
$= 6 + 25\times 3$	
$= 81$	1A

9. Find the sum of all the multiples of 7 (i.e. 7, 14, etc.) that lie between 1 and 100.

求在1與100之間所有7的倍數(即如7,14,等)的總和。

Solution

$7\overline{)100}$ $14....2$	1M (method to consider multiples)
$7 \times 14 = 98$	1A (98 appearing)
$7, 14, \ldots, 98$	1M (recognizing A.P.)
$T_1 \quad T_2 \quad \ldots \quad T_{14}$	1A (14 terms)
$S = \frac{14}{2}(7 + 98)$	1M + 1A
$= 735$	1A

$7\overline{)100}$ $14....2$	1M (method to consider multiples)
$7, 14, \ldots.$	1M (recognizing A.P.)
$T_1 \quad T_2 \quad \ldots \quad T_{14}$	1A (14 terms)
common difference = 7	1A
$S = \frac{14}{2}\left[2\times7 + (14 - 1)\cdot7\right]$	1M + 1A
$= 735$	1A

Write down all multiples and give the sum by direct addition.	4 marks only (no 1P and deduct 1 mark for minor mistakes)

10. An A.P. starts with 17 and ends up with 86. The sum of all these terms is 1236, i.e. $17 + \ldots + 86 = 1236.$ Find the common difference of this A.P.	某算術級數開始是17,最後的是86. 這些項的總和是1236,即 $17 + \ldots + 86 = 1236$。 求此算術級數的公差.

Solution

$$1236 = \frac{n}{2}(17 + 86) \qquad 2A$$

$$1236 = \frac{n}{2} \times 103 \qquad 1M \text{ (solve n)}$$

$$\therefore\ n = 24 \qquad 1A$$

$\therefore\ 86 = 17 + 23d$	$86 - 17 = 23d$	$1236 = \frac{24}{2}(2\times17+23d)$	1M + 1A
$23d = 69$	$23d = 69$	.....	(1M for idea of T_n, diff. or S_n)
$\therefore\ d = 3$	$\therefore\ d = 3$	$\therefore\ d = 3$	1A

$$\begin{cases} 1236 = \frac{n}{2}\ 2\times17 + (n-1)d \ \ldots\ldots\ (1) & 1A \\ 86 = 17 + (n-1)d \ \ldots\ldots\ (2) & 1A \end{cases}$$

From (2), $(n - 1)d = 69$ ....(3) 1A

Sub. in (1),

$$1236 = \frac{n}{2}(34 + 69) \qquad 1M \text{ (elimate d)}$$

.....

$$n = 24 \qquad 1A$$

From (3), $23d = 69$ 1M

$$\therefore\ d = 3 \qquad 1A$$

11. Given an A.P. -10, -3, 4, ...
    Find the following sum:
    3rd term + 6th term + 9th term + ...
    + 24 term

给定一算術級数 -10, -3, 4, …
求以下之和：
第3项 + 第6项 + 第9项 + …
… + 第24项.

Solution

Given A.P. $a = -10$, $d = -3 - (-10) = 7$ 1A

$T_3 = 4$

$T_{24} = (-10) + 23 \times 7$

$= 151$ 1A

$T_3 + T_6 + \ldots + T_{24}$ 1M (as an A.P.)

8 terms together 1A

$\text{Sum} = \frac{8}{2}(4 + 151)$ 1A + 1M

$= 620$ 1A

Given A.P. $a = -10,\ d = 7$ } 1M + 1A

New A.P. $T_1' = T_3 = 4$ (1M for recognizing new A.P.)

$d' = 3d = 21$ or $d' = T_6 - T_3 = 21$ 1A

8 terms by counting or other method 1A

Sum to $T_8' = \frac{8}{2}[2\times4 + (8 - 1)\cdot21]$ 1M + 1A

$= 620$ 1A

Obtain the sum by direct listing of all the terms:

$4 + 25 + 46 + 67 + 88 + 109 + 130 + 151$

$= 620$ 2M + 2A (deduct 1A for minor mistakes; no 1P in this case)

12. The sum of the first 4 terms of an A.P. is 26. The sum of the next 6 terms is 129. Find the sum of the first 20 terms of this A.P.

某算術級數的前4項之和是26，接著的6項之和為129。求此級數前20項之和。

Solution

$S_4 = \frac{4}{2}(2a + 3d) = 26$ 1A

$S_{10} = \frac{10}{2}(2a + 9d) = 26 + 129 = 155$ 1M + 1A (1M for adding)

$\begin{cases} 2a + 3d = 13 \\ 2a + 9d = 31 \end{cases}$ 1M (for solving)

$6d = 18$

$\therefore d = 3$ $\frac{1}{2}$A

$2a + 3\times3 = 13$

$\therefore a = 2$ $\frac{1}{2}$A

$S_{20} = \frac{20}{2}\ 2\times2 + (20 - 1)\cdot3$ 1M (correct formula)

$= 610$ 1A

$$S_4 = \frac{4}{2}(2a + 3d) = 26 \qquad 1A$$

$$S_{10} - S_4 = \frac{10}{2}(2a + 9d) - \frac{4}{2}(2a + 3d) = 129 \qquad 1M + 1A$$

$$\left[\text{Or } \frac{6}{2}\left[(a + 4d) + (a + 9d)\right] = 129 \right]$$

(1M for subtracting or as new A.P.)

$$\begin{cases} 2a + 3d = 13 \\ 2a + 13d = 43 \end{cases} \qquad 1M\text{(for solving)}$$

$$10d = 30$$

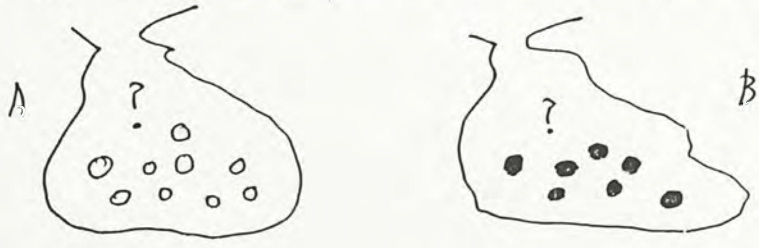
$$\left.\begin{aligned} \therefore \quad d &= 3 \\ a &= 2 \end{aligned}\right\} \qquad 1A$$

$$S_{20} = \frac{20}{2}\left[2\times2 + (20 - 1)\cdot3\right] \qquad 1M \text{ (correct formula)}$$

$$= 610 \qquad 1A$$

13. Bag A contain some white marbles. Bag B contains some black marbles.
3 white marbles are first taken out from Bag A.
6 black marbles are then taken out from Bag B.
9 white marbles are taken out from Bag A in the 3rd trial.
12 black marbles are taken out from Bag B in the 4th trial.
This pattern of taking out matbles from alternate bags is repeated and each time the number of marbles taken is increased by 3.
It is found that after 19 trials, both bags are empty.
Find the number of marbles in each bag.

A袋裝着白色波子若干，B袋裝着黑色波子若干。
先從A袋取出3粒白色波子，
再從B袋取出6粒黑色波子，
第3次由A袋取出9粒白色波子，第4次從B袋取出12粒黑色波子。按此規則繼續由A及B袋交替取出波子，每次均多取3粒。
19次之後知道兩袋皆空，求各袋有波子若干粒。



Solution

A 3 9 15 ...

B 6 12 ... } 1M (clearly separating the 2 A.P.'s)

19 trials ∴ end at A

10 trials from A, 9 trials from B — 1A (correct nos. of terms)

Recognizing the above 2 A.P.'s — 1A (correct a's and d's)

For A, $3 + 9 + 15 + \ldots$ (10 terms)

$= \frac{10}{2}(2 \times 3 + 9 \times 6)$ or $\frac{10}{2}(3 + 57)$ — 1A

$= 300$ — 1A

For B, $6 + 12 + \ldots$ (9 terms)

$= \frac{9}{2}(2 \times 6 + 8 \times 6)$ or $\frac{9}{2}(6 + 54)$ — 1A

$= 270$ — 1A

Alternatively,

total no. $= 3 + 6 + 9 + \ldots$ (19 terms)

$= \frac{19}{2}(2\times3 + 18\times3)$ or $\frac{19}{2}(3 + 57)$ — 1A

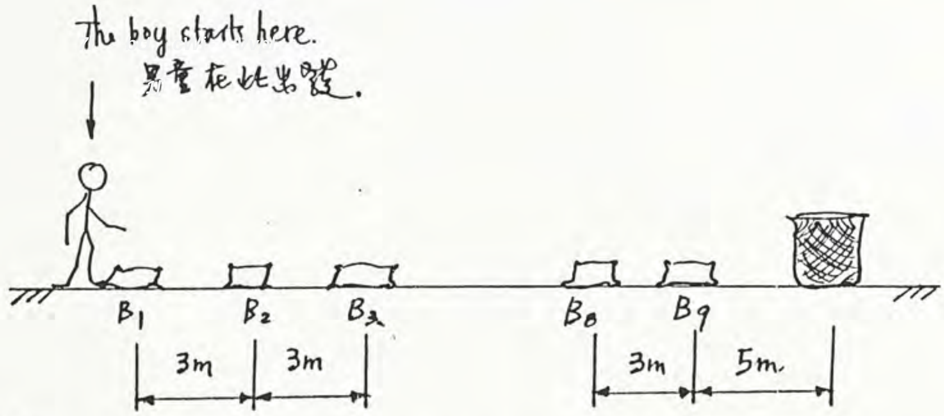
$= 570$ — 1A

Then use this to minus either the number for A or B to get the remaining unknown.

Write down all numbers and get the answers by direct summation. — 4 marks only (no 1P and deduct 1 mark for minor mistakes)

14. In playing a game of picking up sand-bags, a boy is asked to pick up 9 and-bags and put them into a basket, one at a time. The bags are arranged in a straight line and the distance between two consecutive sand-bags is 3 m (see figure). The basket is placed 5 m away from the last sand-bag B_9. The bay starts at B_1, pick up sand-bag B_1 and runs to the basket. Then he returns to pick up B_2 and run to put it in the basket again. This repeats until he puts the last one (i.e. B_9) into the basket. Find the total distance travelled by the boy in this game.

在一拾沙袋的遊戲中，男童須要逐一拾起沙袋共9個，放進一竹籃。沙袋排成一直綫，相鄰沙袋各相距 3m（見圖）。竹籃與最後沙袋 B_9 相距 5m。男童從 B_1 出發，先拾起 B_1，走到竹籃處放下。接着返回 B_2，拾起沙袋 B_2，再奔回竹籃，如此類推，直至將最後一個沙袋（即 B_9）放進竹籃為止。求男童在遊戲中共跑了多少路程。



Solution

$B_1 \longrightarrow$ basket	$24 + 5 = 29$	29	1M (showing understanding of problem statements)
basket $\longrightarrow B_2 \longrightarrow$ basket	$(21+5)\cdot 2 = 26\times 2$	52	
basket $\longrightarrow B_3 \longrightarrow$ basket	$(18+5)\cdot 2 = 23\times 2$	46	1A (identifying some terms of this A.P.)
....			
basket B_9 basket	5×2	10	

Total distance travelled

$= 29 + 2(\underbrace{26 + 23 + \ldots + 5}_{8\text{terms}})$

$= 29 + 2 \times \frac{8}{2}(26 + 5)$

$= 277$ m

$= 29 + (\underbrace{52 + 46 + \ldots + 10}_{8\text{ terms}})$

$= 29 + \frac{8}{2}(52 + 10)$

$= 277$ m

1A (correct terms to be added)
1M (an A.P. adding another number)
1A (correct no. of terms)
1M (for applying formula)
1A

$B_1 \longrightarrow$ basket $\longrightarrow B_2$	$(24+5) + (5+21)$	$= 55$	1M (showing understanding of the problem statements)
$B_2 \longrightarrow$ basket $\longrightarrow B_3$	$26 + 23$	$= 49$	
$B_3 \longrightarrow$ basket $\longrightarrow B_4$	$23 + 20$	$= 43$	1A (identifying some terms of this A.P.)
....			
$B_8 \longrightarrow$ basket $\longrightarrow B_9$	$8 + 5$	$= 13$	
$B_9 \longrightarrow$ basket		5	

$\therefore$ total distance travelled

$= \underbrace{(55 + 49 + 43 + \ldots + 13)}_{8 \text{ terms}} + 5$ — 1A (correct terms to be added); 1M (an A.P. adding another number); 1A (correct no. of terms)

$= \frac{8}{2}(55 + 13) + 5$ — 1M (for applying formula)

$= 277$ m — 1A

Write down all distances travelled and then get the answer by direct addition. — 4 marks only (no 1P and deduct 1 mark for minor mistakes).

END OF TEST 全卷完

Appendix E

Items for Task-Based Interviews

There were 20 items for task-basedinterviews, ordered according to level of difficulty. All items would be presented to the subjects in this order. In some cases, the subjects felt tired or unmotivated to carry on, and the interview stopped before the last item had been presented. This batch of items was designed to test knowledge of various parts of the topic 'arithmetic progressions', some relating to conceptual knowledge, and mostly on procedural knowledge and their application in harder problems which demanded synthesis of these procedures. The concepts or procedures involved in each item are also presented with the item below. These are marked by asterisks beneath the problem statements.

---

1. Explain what is meant by an A.P. (arithmetic progression).

   Please give one example of an A.P. and give one example that is NOT an A.P.

   解釋什麼是算術級数 (A.P.)。

   舉一個 A.P. 的例子。
   舉一個不是 A.P. 的例子。

* Concept of an A.P.

* Other notions related to the concept of an A.P.

2. Indicate whether each of the following sequences is an A.P. or not.
   If yes, state its first term and common difference.
   If no, state your reasons.

   指出下列数列那些是算術級数。
   若是，指出其第一项及公差。
   若不是，請解釋。

   (a) 4, 7, 10, 13, 16

   (b) 12, 14, 17, 21, 26

   (c) -3, 1, 5, 9, 13

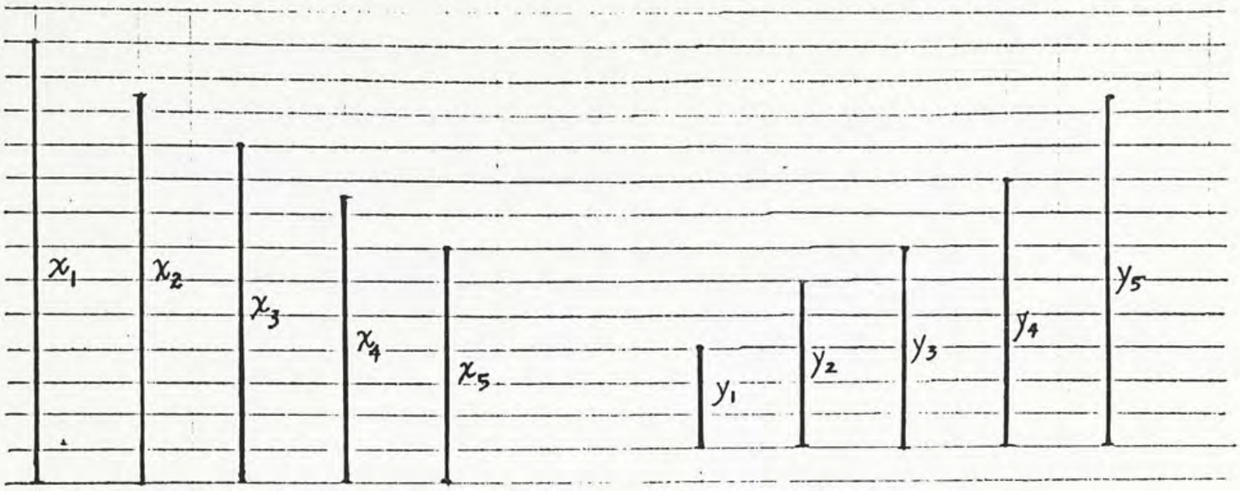
   (d) 4, -7, -18, -29, -40

   (e) x+1, x+3, x+5, x+7, x+9

   (f) 5, 55, 555, 5555, 55555

* Discrimination of an A.P.

* Identification of the first term and the common difference for an A.P.



3. In the figure above, x_1, x_2, ..., x_5 and y_1, y_2, ..., y_5 represent the lengths of the line segments as shown.
State whether each of the following is an A.P. or not. Why or why not ?

上圖中 $x_1, x_2, \cdots, x_5$ 及 $y_1, y_2, \cdots, y_5$ 乃是各綫段的長度。
下列是否算術級数？為什麼？

(a) x_1, x_2, x_3, x_4, x_5

(b) y_1, y_2, y_3, y_4, y_5

* Identification of patterns for A.P.'s as realized by a concrete example
* Association of notions of A.P.'s with corresponding physical quantities
* Application of concepts concerning A.P.'s to a physical situation

4. Given that p, q, r is an A.P., which of the following is/are also in A.P. ? Why or why not ?

已知 p, q, r 是算術級数，
以下那些是算術級数？為什麼？

(a) p+8, q+8, r+8

(b) 8p, 8q, 8r

(c) p^2, q^2, r^2

(d) r, q, p

* Discrimination of A.P.'s in more abstract cases which involve formalistic execution of the definition
(knowledge of algebraic manipulation is involved)
* Use of counterexamples to reject the possibility of an A.P.
* Relation between a given A.P. and another sequence derived from it
* Difference between terms changed by arithmetic operations

5. Given an A.P.

3, 7, 11, 15, 19, ...

(a) Find the sum to three terms.

(b) Find the sum of the 4th term to the 6th term.

(c) If its 20th term is $(5a + 9)$, find the value of a.

(d) If the kth term is 95, find k.

已知某算術級数

3, 7, 11, 15, 19, …

(a) 求加至三項之和。

(b) 求第4項至第6項之和。

(c) 若第20項為 $(5a+9)$，求 a 的值。

(d) 若第 K 項是 95，求 K。

(a) * The concept of 'sum of terms'

* Use of simple arithmetic/ summation formula

(b) * The concept of 'sum of terms not starting from the first'

* Use of simple arithmetic/ summation formula

(c)/(d) * Use of formula to find terms

* Identification of the values for using the formula

(d) * Use of the general term formula as an equation

6. How many terms are there in the following A.P.:

10, 13, 16, ..., 94 ?

以下的算術級数有多少項：

10, 13, 16, …, 94 ?

* Association of the concepts of an A.P. with a presented sequence
* Recognition of the meaning of the number of terms
* Application of the general term formula as an equation to solve n / Use of some representations that give n by arithmetic
* Identification of the values for the procedures

7. If 27, x, y, z, 39 is an A.P., find x, y and z.

若 27, x, y, z, 39 為算術級数，求 x, y 及 z。

* Association of the concepts of an A.P. with a presented sequence
* Use of formula for the general term to the situation/ Use of some representations identify the common difference as the essential unknown
* Any special conceptualization that would facilitate the solution

8. Given an A.P. 3, 9, 15, 21, ...

(a) Write down the nth term.

(b) Find the sum of the first 20 terms.

(c) Find the sum of the 41st term to the 60th term.

(d) Find the following sum:

3rd term + 6th term + 9th term + ... + 15th term + 18th term

已有一算術級数 3, 9, 15, 21, ⋯

(a) 寫出其第n項。

(b) 求前20項之和。

(c) 求第41項至第60項之和。

(d) 求以下之和：

第3項 + 第6項 + 第9項 + ⋯ ⋯ + 第15項 + 第18項

(a) * Use of the general term formula, identifying the knowns and the unknown

(b) * The concept of 'sum of the first .. terms'

* Use of the summation formulas

(c) * The concept of 'sum of the .. term to the .. term'

* Use of the summation formulas with appropriate modification either through conceptualizing two sums or a new A.P. with changed parameters

(d) * The property of a sub-sequence of an A.P.

* Identification of parameters for a new A.P. derived from a given A.P.

* Application of summation formulas to a new A.P. derived from a given A.P.

9. In an A.P., the 11th term is -32, and the 35th term is 64. Find the common difference of this A.P.

某算術級数之第11項為 -32，第35項為 64。求此算術級数之公差。

* The representation of a problem with two terms given

* Use of the general term formula or other relevant procedure

* Use of the general term formula as an equation, taking the first term and the common difference as two essential characteristics of an A.P.

10. In an A.P., the 6th term is 20, and the 11th term is 50. Find the 27th term.

某算術級数之第6項為20，第11項為50。求第27項。

* The representation of a problem with two terms given

* Use of the general term formula or other relevant procedure

* Use of the general term formula as an equation, taking the first term and the common difference as two essential characteristics of an A.P.

* If Problem 9 was solved by an effective representation, can this problem be solved similarly or a different strategy is adopted ?

* Planning the solution in stages, i.e. breaking it into familiar subproblems

11. $(9x + 4)$, $(7x + 2)$, $(3x + 4)$, ... is an A.P.

(a) Find the value of x.

(b) Find the value of the 5th term.

$(9x+4)$，$(7x+2)$，$(3x+4)$，… 乃算術級数。

(a) 求 x 的值。

(b) 求第5项的值。

* The essential property for three numbers in A.P.
* Solving an A.P. by identifying its first term and the common difference
* Use of the general term formula to find terms by substituting the identified values of each parameters

12. Find the sum of the following A.P.:

$8 + 15 + 22 + \ldots + 78$.

求以下算術級数之和：

$8+15+22+\cdots+78$ 。

* Identify the unknowns necessary for the problem
* Breaking the problem into familiar or solvable subproblems
* Use of the general term formula as an equation or other useful representation
* Use of the summation formulas for finding sum given relevant data

13. Given an A.P.

324, 308, 292, 276, ...,

find the first <u>negative</u> term in this A.P.

給出一算術級数：

324，308，292，276，…，

求此級数中第一個<u>負数</u>的項。

* Recognition of a decreasing A.P. associating with a negative common difference
* Use of an effective representation that can suggest ways of solution
* Understanding the relationship between terms and the common difference (formula cannot be applied directly without a correct conceptualization of the situation in terms of ideas of A.P.)

14. Find the sum of all the numbers from 1 to 60 which are NOT divisible by 3.

求1至60之間所有<u>不能</u>被3整除的数字的總和。

* Searching for patterns that give rise to A.P.'s in a situation
* Identify the essentials of the A.P.'s for finding the sums
* Apply one of the summation formulas in an appropriate way

15. An A.P. 42, ... , 133 has a sum of 1225, i.e.

$42 + \dots + 133 = 1225$.

Find the common difference of this A.P.

某算術級数 42,…,133 之和是 1225，即

$42 + \cdots + 133 = 1225$，

求此算術級数之公差。

* Identify the unknowns in this problem
* Breaking the problem into familiar or solvable subproblems
* Use of the summation formula that involves the first and the last terms as an equation
* Use of the general term formula as an equation or other useful representation connecting the number of terms and the common difference

16. An A.P. has a common difference of 4. Its 10th term is three times the 5th term.
Find the sum of the 11th term to the 16th term.

某算術級数之公差是 4，其第 10 項是第 5 項的三倍。求第 11 項至第 16 項之和。

* Identify the unknowns for solving the problem
* Breaking the problem into familiar or solvable subproblems
* Use two general terms to form an equation
* Use of the summation formula to find the sum of an A.P. not starting from the first term (i.e. either by generating a new A.P. or by considering the requiring sum as a difference of two sums)

17. A theatre has x rows of seats. The first row has 28 seats, the second row has 32 seats, the third row 36 seats and so on with an increase of 4 seats per row until the last row has 64 seats.

(a) How many rows are there ?

(b) How many seats are there ?

某戲院有座位 x 行。
第一行有座位 28 個，
第二行有座位 32 個，
第三行有 36 個。如此類推，每行增多 4 個座位，直至最後一行有 64 個。

(a) 共有多少行座位？

(b) 共有多少個座位？

* Representation of a situation in terms of concepts of A.P.'s
* Identification of the unknowns of the problem in terms of the concepts of A.P.'s
* Distinguishing the term and the sum in a situation
* Use of the general term formula and the summation formula for calculation

18. Consider an A.P.

$T_1, T_2, T_3, T_4, \ldots$

Suppose $T_5 = 3$
and $T_{10} = 18$.

(a) If $T_k = 39$, find k.

(b) Find the following sum:

$T_2 + T_4 + T_6 + \ldots + T_{14}$.

考慮一算術級數

$T_1, T_2, T_3, T_4, \cdots$

已知 $T_5 = 3$ 及 $T_{10} = 18$。

(a) 若 $T_K = 39$，求 K。

(b) 求以下之和：

$T_2 + T_4 + T_6 + \cdots + T_{14}$。

(a) * To handle A.P.'s in more algebraic settings
* Representation of a problem with two terms given
* Identification of the essential unknowns of the problem
* Breaking the problem into familiar or solvable subproblems
* Use of the general term formula as an equation

(b) * Recognition of a sequence as an A.P.
* Relation between an A.P. and a sequence derived from an A.P.
* Use of the summation formula for an A.P. derived from a given A.P.

---

19. A student has picked up a broken piece of paper on which there is a multiple choice question. Part of the question has been crossed out.

某學生拾獲一張殘缺紙片，上面有一多項選擇題，但部分問題被塗去。

> 7. [redacted] + 19 + 23 + [redacted]
>
> The above shows an A.P. with 15 terms. Their sum is equal to
>
> 上列算術級數有15項。它們的和是
>
> A. 315
> B. 480
> C. 525
> D. 735
> E. 885

After calculating for a moment, the student has found the correct answer. Do you know which one ?

他計算了一會，終於找到正確答案。你知道是那一個嗎？

* Use of the summation formula that involves the first term, the number of term, and the common difference
* Representation of the problem with relevant parameters
* Use of the trial-and-error method
* Recognition of the condition of consistency to determine the answer

20. Two ants A and B are 54 mm apart. They start at the same time and crawl <u>towards</u> each other along a string (see Figure).
Ant A crawls 0.5 mm in the 1st second, 1.5 mm further in the 2nd second, 2.5 mm further in the 3rd second and so on.
Ant B crawls 1 mm in the 1st second, 3 mm further in the 2nd second, 5 mm further in the 3rd second and so on.

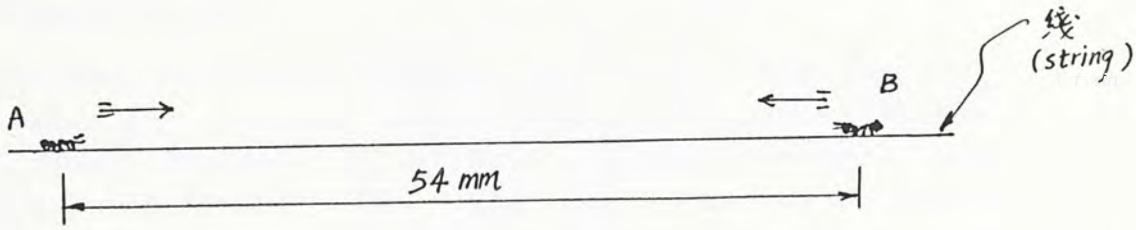
(a) After how many seconds will the two ants meet ?

(b) If they continue to crawl after meeting, what is the distance between them 1 second later ?

兩隻螞蟻A及B相距54 mm。它們同時開始，在一線上<u>相向</u>爬行（見圖）。
螞蟻A在第1秒內爬行0.5 mm，第2秒內再多行1.5 mm，第3秒內再多行2.5 mm，如類推。
螞蟻B在第1秒內爬行1 mm，第2秒內再多行3 mm，第3秒內再多行5 mm，如類推。

(a) 它们多少秒鐘後相遇？

(b) 若相遇後它們仍繼續爬行，1秒後它們會相距多遠？



* Representation of problems in terms of concepts of A.P.'s
* Identification of A.P.'s
* Identification of parameters of A.P.'s
* Identification of unknowns in terms of notions of A.P.'s
* Use of the general term formula and the summation formulas for calculation

Appendix F
Sample Protocols of Two Subjects in Task-based Interviews

In the task-based interviews, the verbalizations made by subjects were in Cantonese together with English mathematical terms. Protocols were originally transcribed from cassette recordings in Chinese. Records of two subjects were translated into English and presented here. One was selected from the good performance group and another from the poor performance group. This would help understand the form of raw data involved in the process of analysing protocols and inferring the knowledge structure as discussed in Chapter 5. The knowledge structures constructed from the protocols presented here actually appear in Figures 6 and 7.

In the following samples, I and S stand for the interviewer and the subject respectively. Dots are added to indicate short periods of silence. Other comments are inserted by square brackets to make these sentences more intelligible. For easy reference, these samples are separated into segments according to the problems the subjects were working on. The items for the interview can be found in Appendix E.

<u>Sample 1</u> Protocols of Subject A08
(Good performance group)

Item 1

I: Explain what is meant by an A.P. (arithmetic progression).

S: A list of numbers, with a common difference in between.

I: What do you mean by "common difference"?

S: That is, the differences between numbers are equal.

I: OK Please give one example of an A.P.

S: 2, 4, 6, 8, 10,..

I: And give one example that is NOT an A.P.

S: 1, 4, 8, 13

I: OK

Item 2

I: You may go on reading the question on your own. But please do remember to speak up.

S: (a) is an A.P. Common difference is 3. First term is 4.

I: Next.

S: (b) is not.

I: Why?

S: The 1st and the 2nd terms differ by 2. The 2nd term and the 3rd term differ by 3.

I: OK Next.

S: (c) is an A.P. Common difference is 4 and first term is -3.

I: (d) then.

S: (d) is not.

I: Please explain.

S: The 1st and the 2nd terms differ by -11. The 2nd term and the 3rd term differ by -7.... Hm.. No. Not -7. It's -11 again.

I: What do you mean?

S: Well, it's an A.P...... First term is 4 and common difference is -11.

I: Next is (e).

S: (e) is an A.P. First term is x + 1. ..Common difference +2.

I: Let's come to (f).

S: (f) is not.

I: Can you explain?

S: The 1st term differs from the 2nd by 50. The 2nd and the 3rd differs by 500.

Item 3

I: This is item 3. Read the question.

S: [Read the question.] ....(a) is an A.P. but (b) is not.

I: Why did you say so?

S: These segments in this order differ in lengths by 1.5 units.

I: Why (b) is not?

S: The 1st and the 2nd differ by 2 units and the 2nd and the 3rd differ by 1.

I: OK

Item 4

S: [Read the printed question.]... p, q, r is an A.P. Now each is added to 8....yes, an A.P.

I: Why?

S: The differences between them are not affected by adding 8 to all of them.

I: How about (b)?

S: (b) is [an A.P.] since they are all multiplied by 8, their differences will be equal.

I: Next.

S: (c) is not [an A.P.]

I: Can you explain?

S: For example, you put p to be 1, q be 2 and r be 3. Then p squared is 1, q squared is 4 and r squared is 9. [She meant that this would not form an A.P. then.]

I: Next.

S: (d) is [an A.P.]

I: Why?

S: .... p, q, r is an A.P. Now they are reversed in order. That is r, q, p. Therefore it should also be [an A.P.] That is the common difference becomes negative.

I: OK.

Item 5

S: [Read the question.] an A.P., 3, 7,...(a).. sum to three terms ..consider the difference, 7 minus 3, 4, 4 times 2 [writing down expressions] 8,. ..14, ...3, 7, 21.. the answer is 21.
Part(b),..4th to 6th term,.Hmm..., two d's, Hmm.. d is 2. ..No, d is 4. 8, 15 plus 8....[writing expressions] 23..[using calculator] 34.5 is the answer.

I: Would you explain your expression here?

S: This is the formula n[2a+(n-1)d]/2.

I: What are a and d here?

S: a is the 4th term.

I: d is ...

S: the common difference which is 4.

I: Go on.

S: Part(c). 20[th term] is 5a + 9... 5a plus .. 3 plus.. 19 [d's] d is 4... 82 plus 3, 5a + 9 so 85 minus 9 is equal to 5a, 74 [use calculator].. why decimals? .. wrong ... 3 plus 19 times 4 ..[use calculator] should be 76 .... [repeat the former equation] yes..a is 14. [Part (d)] Hmm...k is 95, k, 95.... 3 plus (k - 1) times 4 [writing this equation] is 3 plus 4k minus 4 .. so 96 is 4k. k is 24. [by mental calculation]

Item 6

S: [Read the question.]..... 94 is equal to 10 plus (n - 1) 3's [writing an equation] that is, 94 equals 10 plus 3n minus 3, 84, 10 plus 3n,....

3n minus 3. 81 is 3n so n is 81 divided by 3, equals ..6..7..27. n is 27. [by mental calculation]

I: You have written down n here. But what is it actually?

S: n is the number of terms.

I: You mean that is your answer.

S: Yes.

Item 7

S: [Read the statement.] in A.P... [Point to the space between the numbers and 3. unknowns and count.] one, two, .., four. 4d is 39 minus 27. [Write the equation.] d is 3. So x is 27 + 3 = 30.
y is 30 + 3 = 33. z is 33 + 3 = 36.

I: How did the number 4 come about in your calculation.

S: There are 5 terms here....You see the common differences are separated into four parts.

I: What do you mean by "separated into four parts"?

S: That is four common differences in between.

Item 8

S: Given A.P...the nth term is ..3 plus (n - 1) .... 6's [writing the correct algebraic expression] [continue immediately to (b)] 20th term is 3 plus 19 times 6.
.... no.. sum of 20 terms.. should be [write 20]
2 times 3 plus 19 times 6 [and then] over 2 ...
10 times 2 , 3 be 6 plus 114 is 120 cross 10 is 1200.
[answer] Part(c) find the sum from 41st to 60th term
T 41 [i.e. 41st term] is ... 3 plus 40 times 6 ..
Hmmm.. 243. T 60 is 3 plus 59 times 6 [use calculator]
357. Hmmm. S is 1 over 2, 40, 60 mius 41...[doubting]
.....[pause] 243 plus 357 [use calculator] is
10 times 600, is 6000.

I: OK I observe that when you wrote 60 minus 41 and got 19, why did you change it to 20?

S: I count also 41 [st term] then.

I: Can you explain why you can do this?

S: This is the summation formula using the first term and the last term. Here, we got them.

I: So you use it.

S: Yes.

I: You may go to part(d).

S: The 3rd term, the 6th term, ..., yes, an A.P. again There are .. 6 terms altogether. The new common difference is greater... should be 3d, ... i.e. 3 times

6 , 18,.. The 3rd term is 15 [given] So the sum is [write down the second summation formula that involves a, d and n] 2a, a is 15 plus 5 times 18 which is 3 times (30 plus 90), 120, 360. S is 360.

Item 9

S: [Read the question.] 11 is -32, 35, 64....
common difference, T 11 minus ... T 35 minus T 11
is -32 minus 64 is.... 96 [pause] NEGATIVE 96 ....

I: Please talk.

S: -32, 64... the 11th term is less than the 35th term
That is the value should be positive..but now it's negative...Hmmm.. Well..Aha. I've copied them in the wrong order, yes, 96 over 24 is ... 4, that is the common difference.

I: May I ask you what does 24 refer to?

S: There are 24 terms.

I: Which 24 terms?

S: No. With 24 differences in between.

Item 10

S: [Read the question.] O.K.
The 6th term is 20, the 11th is 50.
Then 5d is 30 [by mental computation 50 - 20].
.. d is 6. [Write down d = 6.]
That is, 27 minus 6 is 21.
[Write down T_{21}.]
equals a .. a is 20 [This is T_6.] ..
plus 20 [this is (21 - 1)] times 6 ..
which is 120 plus 20 is 140.
[The following expression is written
$T_{21} = 20 + 20 \times 6 = 120 + 20 = 140$.]
So, T_{27} is 140.

I: Why did you say a is 20?

S: This [pointing to T_6] can be treated as the first term for calculation.

I: Then....

S: The 27th term to be found would be the 21st term in this new A.P.

I: I see. So you write T_{21} here?

S: Yes.

Item 11

S: [Read the question.] So find x...that is to say

9x minus ... 9x + 4. Hmm.. Yes 7x + 2 minus 9x + 4 is equal to 3x + 4 minus 7x + 2 [write down an equation] .... $7x + 2 - 9x - 4 = 3x + 4 - 7x - 2$ [solve the equation] x is 2

I: What does the first equation mean?

S: This is the difference between terms. They are equal.

I: OK You may go on.

S: Part(b) the 5th term. 9x + 4, x is 2 so 18 + 4, 22 T 1 is 22. Hmmm. 14 plus 2 is 16. 22 minus 16 is 6. T 5 is a, a?. a is 22, that is ..No. 16 minus 22 is -6. 22 plus four .. four -6's. 22 plus 6, 4, 24, -24. T 5 is -2.

Item 12

S: [Read the question.] [Pause.] .. 15 minus 8 is 7. ...... 78 is .... 8 plus (n - 1) 7's, that is 8 + 7n - 7, 78, 70 equals 7n - 7. 63 is 7n. n is 9. And S 9 [meaning sum of 9 terms] is 1 over 2 times ...... two 8's plus (n - 1) times 7, 9 minus 1, 8, 8 times 7 [use calculator] 324.

I: What did you want to get for this first line?

S: To find out how many terms.

I: Is it necessary? Are you sure you need this?

S: Yes.

I: You used to do mental calculation and seldom use the calculator. Is it your common practice?

S: Yes, I usually do simple arithmetic mentally.

Item 13

S: [Read the question.] the first negative term...quite difficult I think...Hmmm.[Murmuring.]..324 minus 308 is 16. 324 divided by 16 [use calculator] 20.25 that is..20...21. The 21st term is negative.

I: Can you explain what you are doing?

S: Now common difference is 16. The first number is 324. I'll try to see how many differences 324 can have.

I: You have said 20 and 21. How do you know which is the answer?

S: I think the 20th number is still positive and so it is the 21st.

Item 14

S: [Read the question.] [Quite immediately.] S 60 [meaning sum of 60 terms] is half 60 times 2a, 2a that is 1, ..

..no. 2a is 2 plus (n - 1), 59 1's, equal to 61, times 61 is 1830. Let's see. 60.... 20. There are 20 [numbers]. S 20 is 20 over 2, 6 plus 19 times 3... [simplifying an expression] 630. 1830 minus 630 is 1200.

I: Can you explain your steps?

S: I add up all numbers first here. Then minus all those multiples of three. And there 20 multiples of 3 inside.

I: You have a clear plan when you first added all the numbers?

S: Yes. I know I am going to deduct.

Item 15

S: [Read the statements.] 42, 133, common difference... 1225 is half .... x? times 42 plus 133. [Write an equation and solve.] So x is 14.... 133 minus 42 .. 91 [pause] ..what then?.... 14 terms...1225.. find what? [She substituted 14 into the summation formula to get the given sum again!]..Aha it's d. a is 42. ... plus ..plus, 133...Hmmm that is the 14th term. 13 d's ..[voice too soft/but writing the general term formula to get an equation in d] .. d is 7

I: May I ask you why did you try 133 minus 42 getting 91 before?

S: I tried 91 divided by 14 but that is not an integer. So I doubt.

I: Well when you write x here. What did you think was x?

S: The formula fits in but with one unknown. I know it's n. But I just forget it when I arrive at 14. I don't know why I would forget so important a symbol.

Item 16

S: The 10th term is three times the 5th term..That is a + 9d is ... 3 times a + ... 4d [write this equation] So a + 9d is 3a + 12 d. 2a is ... d is 4 so [solve the equation] a is -4..... Now 11th to 16th terms, 11, T 11 is -6 plus 10 times 4, 36, T 16 minus T 11 is 5. .. so half times 6...... T 16 is -6 plus 15 times 4. -6 plus 60 is 54. Half ..hmmm. 11, 12, 13, 14, 15, 16, 6 terms, times 36 plus 54...... 90 times 3 so 270.

I: I observe that you can write the first equation almost immediately. What did you want to do at the first moment?

S: To find a and then try.

I: You know already the later steps here?
S: I don't exactly know. But I think that would be OK when both a and d are known.

Item 17

S: [Read the question. At the same time, she wrote the list 28, 32, 36, ________ 64 on paper.] .... So 64 is 28 plus (n - 1) times 4. [Write an equation and then solve.] .... n is 8. 8 rows.
I: What is this n?
S: I take it as the number of rows.
I: OK You may go on.
S: For (b)... that is S 8. Equal to 8 over 2 times.. Hmmm a, a? a, 28, 2a plus 7 d's. d is 4. 4 times 56 plus 7, 4, 28, 4 times 48 is 192. There are 192 seats.

Item 18

S: T 1, T 2, T 3, T 4, T 5 is 3. T 10 is 18...T k is 39. k? [Pause.] T 10 is 18. T 5 is 3. 5, 15, difference is 3. What is a? Difference is 3...T k is 39. T k.... equals 39.... k is k ...... T k minus 5 is ...... minus 5 is... T 1 is ..[pause] 3, a,... Aha T 5 is a plus 4d and T 5 is 3 equal to 4d. d is 3. 12, 3 minus 12, a is negative.... -9. Yes, T k is 39. 39 is a, Hmm a, -9 plus ... (k - 1) d's d is 3. That is, -9 plus 3k minus 3. [Solve the equation.] k is 17.
I: Well, I want to know what you meant by T k minus 5.
S: I want to construct a new A.P.
I: So, it means ... Why 5?
S: It's T FIVE. T 5 is 3, given!
I: Thank you. You may go to part (b).
S: T 2 plus T 4 ..... difference is.. changed to 6. Now the new difference. Add up to 14, that is 7 terms. S 7 is, T 2 is -9 plus 3 equals -6. -6 .... 7 over 2 times -6 .... plus n .. 6 d's, d is 6... [use calculator] 72.

Item 19

S: [Read the question.] Their sum is.. only one is correct? ....Hmmm, 19, we don't know the first and the last terms....very difficult... 23 minus 19 is 4.

The difference is 4. ... It would take a lot of time. ... I don't know.

I: Please try a bit longer.

S: [Pause.] The difference is 4... difference is 4. .... We don't know which term it is. Try these one by one? ... It would be too long. ...Hmm. Really difficult..... It is not given which terms these are. Here are addition signs. So, [19 is] the 14th term at most. Can I write down the 13 terms before them? ... . Very difficult. I want to give up.

I: OK Thank you. Let's try next.

Item 20

S: [Read the question slowly.] WHEN will they meet? Ah...[seemed tired] [pause] WHEN? ....
Add up together? That is n over 2 times a, is 0.5, d is 1, plus.... (n - 1) 1's .... is 2 over n [spoken wrongly, correct written form]
0.1 times 2 plus common difference is 2, (n - 1) times 2..... [write an equation, equating the two.]
But? WHEN? Travel so much in the 1st second. So much in the 2nd. ..... That is, this A, no this B always travels faster than A one more time. [She meant twice.] To go on like this, I doubt!.. I think this is wrong. This is rather difficult. WHEN will they meet? Can I stop here?

I: May I ask you why you equated the two sums here?

S: Hmm. They will meet when these two values are equal. I mean they travel the same distance at the nth second when they meet.

I: So this is the equation for it.

S: I think so.

I: Thank you very much. Let's stop here.

Sample 2 Protocols of Subject A12
(Poor performance group)

Item 1

I: Please explain what is meant by an A.P. (arithmetic progression).

S: That is, all numbers followed one and other.

I: What do you mean by "followed"?

S: The same.

I: All the numbers are the same.

S: No. I mean they are separated by the same difference.

I: What do you mean by "separated"?

S: ..... That is... just like numbers with difference

I: OK Would you please give one example of an A.P.?

S: 1, 3, 5, 7, 9

I: Yes. Give one example that is NOT an A.P.

S: 1, 2, 3, 4

I: You mean this is NOT an A.P.?

S: ...2, 3, 5, 6

I: You want to give a new one or change the first one.

S: I think this is not an A.P.

I: By the way, in the following session, I would not tell you whether your answer is correct or not. You just work on your own. That is enough. When I say "yes", I do not necessarily mean that your answer is correct. I only keep you going. You see.

S: Yes.

Item 2

I: Now, you may go on reading the question on your own. Please do remember to speak up.

S: (a) is an A.P.

I: 1st term is..

S: 4

I: Common difference?

S: 3

I: Next.

S: (b) is not.

I: Why?

S: ...These differ by 2 [pointing to the first two numbers]. These differ by 3 [the 2nd and 3rd numbers].

I: OK Next.

S: (c) is an A.P.

I: First term is..

S: -3

I: Common difference..
S: Differ by 4.
I: We come to (d) then.
S: [Pause.] ....(d) is not.
I: Please explain. What do you find?
S: [Pause.]
I: Just try.
S: Because the numbers are almost all negative except the first one.
I: Next is (e).
S: Yes. An A.P.
I: First term is?
S: x + 1.
I: Common difference is..
S: 2.
I: Part (f).
S: (f) is an A.P.
I: First term is?
S: 5.
I: Common difference...
S: 10.
I: How do you get 10?
S: You get one more number each time. It's 10 times.

Item 3

I: We come to item 3. Read the question.
S: [Read the question.]
I: Which are A.P.'s?
S: Both of them.
I: Why?
S: They all differ by 3 [point to the interval in the horizontal direction].
I: All differ by 3?
S: Yes. Common difference.

Item 4

S: [Read the printed question.]... Hmm. (a) is an A.P.
I: Why?
S: Since the symbols [p, q, r] are still in order.
I: In order? What order?
S: Alphabetical order.
I: OK How about (b)?
S: I don't know.
I: You mean you cannot determine.

S: Yes.
I: OK Next.
S: (c) is not [an A.P.]
I: Why not?
S: English alphabet not in order.
I: Not in order in this case? What order?
S: Is an A.P. Yes, an A.P.
I: OK Part(d).
S: (d) is not [an A.P.]
I: Why?
S: The order is reversed now.
I: May I ask you something more about order?
What do you mean by "order" exactly?
S: ....
I: For instance, now p, q, r is given as an A.P.
What in mind do you have that are A.P.'s?
S: a, b, c
I: Thank you.

Item 5

S: [Read the question.] Sum of the three terms is adding. [Use calculator to add 3, 7, 11.] 21
I: You may continue. I'll not disturb you.
S: Part(b), the 4th term and the 6th term, sum [use calculator] 36
I: How do you arrive at this value?
S: The term following 19 is .... 21. The 4th term is 15. The 6th term is 21. So add them up to give this answer.
I: OK Continue.
S: [Murmuring.]
I: Please do speak up.
S: I don't know this. Find the 20th term. 5a + 9. Is it 5a + 9 equals 20? .. I don't know.
I: OK Try (d).
S: The kth term is 95, find k....[Pause.]
The kth term is, the 1st term is 3, 4th term is 15. But what then? Count to 95? ..... So how to find k? .... May I stop?
I: You really don't want to try. OK Let's go on to the next one.

Item 6

S: [Read the question.] An A.P.... common difference is 3. ...... how many terms? ... 10, 13, 16, ..., 94 [Pause.] Can I list them here? [Repeat the question to the interviewer.]

I: You are free to try any method you think possible. There is no restrictions.

S: Well. 10, 13,...[writing down the numbers in all, for the first time error at 82 and cannot reach 94, the second time, find the mistake, protocol only contains these numbers, so omitted] [Count the number of numbers.] Should be 29 terms.

Item 7

S: [Read the question.] .. 39 minus 27 ...Hmmm. Should I divide it by 4? 3. x, y, z are all 3. Each to add 3..... 27 + 3 is 30. y is 33. z is 36. Yes.

I: Why do you divide by 4?

S: To find the difference between these numbers.

Item 8

S: [Read the question.] What is the nth term?? What is the meaning of this question? What to write down? I give up this part(a). I don't know.

I: Well, you may try part (b).

S: (b) to add the first 20 terms.... Can I just write them down.... 3, 9, 15, 21, 27, 33, [really list down these numbers and use the calculator to add, but wrong answer] [Omit the numbers here.]
(c) Find the sum from the 41st to the 60th term. Too big. I think I would just give it up. .. We can do the same thing for this problem.

I: Do you think you can try other method?

S: No.

I: OK You may proceed to the next part.

S: Part (d).. the 3rd term plus the 6th term plus... An A.P...but how to add. The 3rd term is 15. The 6th term is 33 [from above] ...Hmmm.. I just don't want to continue... All the parts are the same. You can write down the numbers and then add.

I: OK Let's stop here. But why do you think this is an A.P.

S: These numbers [refer to the suffixes] are multiples of 3.

I: Yes, these are multiples of 3. But that means ...

S: These show a regular pattern.

I: What pattern?

S: ... even or odd numbers or multiples

I: How about T_3, T_8, T_{13}, T_{18} [writing down] ?
Is it an A.P.?

S: .... of course not.

Item 9

S: The 11th term is -32. The 35th term is 64.
Common difference.... 64 minus -32, common difference ......yes 35 minus 11 is 24. So 96 divided by 24.
But, what is this expression? ..[Pause.]
I remember something like this. But I don't know how to go on. ... [Long pause.] [Stop eventually.]

Item 10

S: [Read the question.] Find the 27th term. What is a?
T n is a plus (n - 1) d's. T 27 is a plus (27 - 1) d's. 6th term is 20.... 11th term is 50. Difference is 30. Common difference is, is it .... 11 minus 6 is 5. Is it 6? .... Or ....
[Pause.] a and d are not known. I can't find them.
.... So.... No. [Looked confused.]..[Wanted to withdraw again.] [Finally stop.]

I: You have not used the formula you just write down here before. Why do you use it now?

S: I cannot remember it in the beginning. But when I think of the 27th term, I seem to remember something like this.

I: You have always subtracted terms. What would you actually get?

S: I am not sure. But it must be connected with the difference, I mean, the common difference. I just can't remember.

I: Well. You have already given up a few problems. Do you want to take a rest before the next items.

S: No. We just continue.

Item 11

S: [Read the question.] ..Why? The common difference seems to be different. [She refers to the three given expressions.] They are not A.P....
[Pause.] ...... Hmm. [Tired.]
I cannot see any pattern in these numbers.

9 with 4, 7 with 2 and 3 with 4....[Murmuring.]
[Stop after a long pause.]

I: Do you know what is x here?

S: No. Its value is not known.

I: Do you think you can get an A.P. by substituting any value for x here?

S: Hmmm. ....I don't understanding your question.

I: OK Thank you. Let's go to the next one.

Item 12

S: [Read the question.] Add from 8 to 78.
Get the sum. .. 8, 15, difference 7, ...
......8, 15, 22, [Use the calculator to add them.] [But reach 80 instead of 78. Error with the listing of numbers.]
[Get a wrong answer for the sum.]

I: Well. You see. Although we can add up A.P.'s directly, it's likely that mistakes would be made when the number of terms is great. So that is the use of formulas. But you have not used a formula up to now. Can you recall the formula?

S: I remember something like this.
[Write down $S_n = n(2a - 1)d/2$.]
But, I think this is probably wrong.

I: Why do you think it is wrong?

S: I used it in the test yesterday [the written problem-solving test]. But sometimes the answers contain decimals. So I just don't think it is right.

I: So you don't use it today?

S: ...Yes..

I: OK Let's go on.

Item 13

S: [Read the question.] 324, 308,.. decreasing ....
...the common difference is [using calculator] 16... each term minus 16.... [start to work on the calculator by deducting 16 successively]
The answer is -12.

I: Do you mean this is the first negative term?

S: Yes.

I: Are you sure of this answer?

S: I just find it by direct subtraction. Maybe I have made a mistake again. I can't guarantee.

Item 14

S: [Read the question.] Exclude the multiples of 3. Add up the numbers. 1, 2, 3, 4, until 60. Well .. 1 plus 60 is 61. 2 plus... 59 is 61. 3 plus ... 58 is 61. [These numbers are written down and join up by lines when she adds.] .....But exclude the multiples of 3, 3 is not added.... Ah .. 60 is not added.... So ... How can we add? .. 1 is added to 59. 60. 2 added to 58 is 60. Aha. 4 is added to.... [Pause.] I do not remember the trick here. I know there is a trick that works at this point. But I forget it now. ...[Cannot pair up numbers correctly again. The linking becomes random.]
....[Stop after unsuccessful trials.]

I: Well. Here you pair up the numbers. How do you doscover this?

S: That is taught in class. When we have A.P.'s, I remember that we can do this and the sum of these pairs are equal.

I: What use is this?

S: There's a trick to use it. But I can't recover it now.

Item 15

S: [Read the question.] ... Again sum of A.P. 133 minus 42 .... that's useless. Common difference. We just can't start to list the numbers. The second term is not given. .... How can we get numbers with this total? .... I don't think I can. [Stop.]

I: Maybe we try one or two more items before we stop for today.

S: Hmmm. [Appeared to be bored.]

Item 16

S: [Read the question.] A.P.... common difference..4 ...three times the 5th term. Find the sum of the 11th term to the 16th term... Common difference is 4. Difference between 16th term and 11th term..... No...find the sum..[Pause.]
I just can't try anything. I don't know even one term. .... It's very difficult. I certainly can't solve it.

I: Well. Let's try the last item then.

Item 17

S: [Read the question.] ..every row increases 4 seats. The 2nd row is 32. ...every row...so on.... Adding this we have the 64 seats in the last row. [Start writing down numbers.] 36, 40, 44, 48, 52, 56, 60, 64... One two three...[counting silently] ten. 10 rows. The total number of seats is by adding these numbers. [Use calculator to add all these numbers.] There are 460 seats.

I: Can you use other methods to solve this problem apart from direct counting and addition?

S: I cannot remember the correct form for summation.

I: How about the number of rows?

S: Hmm. It's straightforward here.

I: If there are many rows, would you count?

S: I think we can still count. But it would be time consuming.

I: OK Thank you very much for coming to this interview.

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